Magneto-resistance in three-dimensional composites.

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Abstract

In this paper we study the magneto-resistance, *i.e.* the second-order term of the resistivity perturbed by a low magnetic field, of a three-dimensional composite material. Extending the two-dimensional periodic framework of [4], it is proved through a H-convergence approach that the dissipation energy induced by the effective magneto-resistance is greater or equal to the average of the dissipation energy induced by the magneto-resistance in each phase of the composite. This inequality validates for a composite material the Kohler law which is known for a homogeneous conductor. The case of equality is shown to be very sensitive to the magnetic field orientation. We illustrate the result with layered and columnar periodic structures.

Keywords: Hall effect, homogenization, magneto-resistance, magneto-transport.

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1 Introduction

In a conductor with a matrix-valued resistivity ρ, a low magnetic field $h \in \mathbb{R}^3$ induces a perturbed resistivity $\rho(h)$. Due to Onsager relations (see [14, 19]), the perturbed resistivity satisfies

$$\rho(h) = \rho(-h)^{\mathrm{T}}. \tag{1.1}$$

As a consequence, the perturbed resistivity admits the following second-order expansion (see Section 2):

$$\rho(h) = \rho(0) + \mathscr{R}(h) + \mathscr{M}(h,h) + o(|h|^2), \tag{1.2}$$

where $\rho(0)$, $\mathscr{M}(h,h)$ are symmetric matrices and $\mathscr{R}(h)$ is an antisymmetric matrix. On the one hand, according to the Hall effect (see, *e.g.*, [14]), the magnetic field induces a transversal electric field $E_t(h)$ which balances the magnetic force acting on the charge carrier and is perpendicular to the current j. It is given by

$$E_t(h) = \mathscr{R}(h)\, j \perp j, \tag{1.3}$$

where $\mathscr{R}(h)$ is the Hall tensor which reduces to $r(j \times h)$ in the isotropic case. On the other hand, the so-called magneto-resistance $\mathscr{M}(h,h)$ measures the difference between the perturbed dissipation energy and the unperturbed one, namely

$$\rho(h)j \cdot j - \rho(0)j \cdot j = \mathscr{M}(h,h)j \cdot j + o(|h|^2), \tag{1.4}$$

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in which the Hall term plays no role (due to the antisymmetry of $\mathscr{R}(h)$). Expansion (1.4) has to be regarded in connection to the Kohler law [13] which states that the symmetrized resistivity (without the Hall term) $\rho_s(h)$ of a homogeneous conductor satisfies the asymptotic

$$\rho_s(h) - \rho(0) \underset{h\to 0}{\approx} m\,|h|^2 \quad \text{with } m > 0, \tag{1.5}$$

which corresponds to an increase of the magneto-resistance.

When the conductor has a microstructure characterized by a scale ε, the resistivity $\rho_\varepsilon(h)$ depends on the two parameters ε, h. In the framework of the Murat Tartar H-convergence theory (see Section 2 and [17, 18]), the conductivity $\sigma_\varepsilon(h) = \rho_\varepsilon(h)^{-1}$ H-converges to the effective (or homogenized) conductivity $\sigma_*(h)$. Under appropriate regularity conditions for $\sigma_\varepsilon(h)$ (see (2.2)), the effective resistivity $\rho_*(h) = \sigma_*(h)^{-1}$ also satisfies equality (1.1) and the second-order expansion

$$\rho_*(h) = \rho_*(0) + \mathscr{R}_*(h) + \mathscr{M}_*(h,h) + o(|h|^2), \tag{1.6}$$

where $\mathscr{R}_*$ is the effective Hall tensor and $\mathscr{M}_*$ is the effective magneto-resistance tensor.

In his seminal work [2], Bergman gave for a periodic composite material an expression of the effective Hall matrix in terms of the local Hall matrix and the local current fields obtained in the absence of a magnetic field. Bergman's approach was extended in dimension two [7] and in dimension three [8] in the non-periodic framework of H-convergence.

In dimension two, the conductor lies in a plane (e_1, e_2) embedded in a transversal magnetic field $h\,e_3$, so that the local/effective Hall coefficient $r_{\varepsilon/*}$ and the local/effective magneto-resistance matrix $M_{\varepsilon/*}$ are defined by

$$\mathscr{R}_{\varepsilon/*}(h) = r_{\varepsilon/*}\,h \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \quad \text{and} \quad \mathscr{M}_{\varepsilon/*}(h,h) = h^2\, M_{\varepsilon/*}. \tag{1.7}$$

In the periodic case, *i.e.* when $r_\varepsilon(x) = r(x/\varepsilon)$ and $M_\varepsilon(x) = M(x/\varepsilon)$ are oscillating functions of the fast variable x/ε, it was proved in [4] that

$$M_*\,\langle j\rangle \cdot \langle j\rangle - \langle M\,j \cdot j\rangle \geq 0, \tag{1.8}$$

for any unperturbed current j, and that (1.8) is an equality if and only if the Hall coefficient is a constant. By the Kohler law (1.5), the magneto-resistance in each phase satisfies $M = \mu\, I_2$ with $\mu > 0$, which implies the positivity of M_* by (1.8). Then, the positivity of m in (1.5) corresponds to the positivity of M_* in (1.8). Therefore, the inequality (1.8) extends the classical Kohler law (1.5) to anisotropic two-dimensional composites (see [4], Remark 2.6).

This paper extends the results of [4] to three-dimensional composites. In dimension three, the local/effective Hall tensor reads as

$$\mathscr{R}_{\varepsilon/*}\cdot h = \mathscr{E}(R_{\varepsilon/*}\,h), \quad \text{with } \mathscr{E}(\eta) := \begin{pmatrix} 0 & -\eta_1 & \eta_2 \\ \eta_1 & 0 & -\eta_3 \\ -\eta_2 & \eta_3 & 0 \end{pmatrix}, \tag{1.9}$$

where $R_{\varepsilon/*}$ is called the local/effective Hall matrix. First, we obtain a general expression (see Theorem 2.1) for the difference between the effective dissipation energy due to the magneto-resistance and the average of the local dissipation energy. Then, extending a classical duality principle (see Lemma 3.1), we prove that this difference is non-negative (see Theorem 3.1), and equal to zero if and only if the Hall matrix satisfies some compactness condition. In the periodic case, this reads as (see Corollary 3.1)

$$D(h,h) := \mathscr{M}_*(h,h)\,\langle j\rangle \cdot \langle j\rangle - \langle \mathscr{M}(h,h)\,j \cdot j\rangle \geq 0, \tag{1.10}$$

for any unperturbed current field j. Moreover, (1.10) is an equality if and only if

$$\mathrm{Curl}\,(\mathscr{E}(R\,h)j) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^3). \tag{1.11}$$

We also investigate the behaviour of higher even-order terms. We show that inequality (1.10) reverses when the magneto-resistance is replaced by the fourth-order term and the Hall matrix is assumed to be zero (see Proposition (3.1)). However, the equivalent of $D(h, h)$ for even-order term higher or equal to 4 may have both a positive and a negative eigenvalue, so that (1.10) cannot be extended (see Proposition 3.2).

Then, the condition of equality (1.11) is discussed in the case of columnar composites. First, an explicit formula for the difference of dissipation energies $D(h, h)$ (1.10) is given (see Proposition 4.1) for a periodic material which is layered in some direction ξ. Second, for a general columnar structure in the direction e_3, the equality $D(h, h) = 0$ is shown to be very sensitive to the orientation of the magnetic field (see Proposition 4.2). More precisely, the equality $D(h, h) = 0$ implies that $\sigma(y_1, y_2)$ is a tensor product of type $f(h_1y_1 + h_2y_2)\, g(h_2y_1 - h_1y_2)$. For example, in the case of a four-phase checkerboard $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (see figure 4.1 below), we obtain that for any magnetic field $h \neq 0$ perpendicular to e_3, $D(h, h) \neq 0$ if $\alpha_1\, \alpha_3 \neq \alpha_2\, \alpha_4$ (see Proposition 4.3).

The paper is organized as follows. In Section 2, we recall results on the homogenization of the Hall effect and the magneto-resistance in order to establish an asymptotic formula for the effective magneto-resistance. In Section 3, we prove inequality (1.10) and deal with the case of higher-order terms. Section 4 is devoted to the case of equality for layered and columnar composites.

Notations

- $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^d$ for any positive integer d and $(e_1, \ldots, e_d)$ the canonic basis of $\mathbb{R}^d$.
- $\times$ denotes the cross product and $\otimes$ the tensor product in $\mathbb{R}^3$.
- $\mathbb{R}^{d\times d}$ denotes the set of the real-valued $(d\times d)$ matrices and I_d denotes the unit matrix of $\mathbb{R}^{d\times d}$.
- $\mathbb{R}^{d\times d}_a$ (resp. $\mathbb{R}^{d\times d}_s$) is the set of the real-valued $(d\times d)$ antisymmetric matrices (resp. symmetric matrices).
- A^s denotes the symmetric part of A, A^{T} its transposed matrix, and $\mathrm{Cof}(A)$ its cofactors matrix.
- Ω denotes a bounded open set of $\mathbb{R}^d$.
- For $\alpha, \beta > 0$, $\mathcal{M}(\alpha, \beta; \Omega)$ denotes the set of the invertible matrix-valued functions A measurable in Ω and such that
$$\forall\, \xi \in \mathbb{R}^d, \quad A(x)\xi\cdot\xi \geq \alpha|\xi|^2 \quad \text{and} \quad A(x)^{-1}\xi\cdot\xi \geq \beta^{-1}|\xi|^2, \quad \text{a.e. } x \in \Omega. \tag{1.12}$$
- For a vector-valued function $U : \Omega \longrightarrow \mathbb{R}^d$,
$$DU := \left[\frac{\partial U_j}{\partial x_i}\right]_{1\leq i,j\leq d}, \quad \mathrm{div}\,(U) = \sum_{i=1}^{d}\frac{\partial U_i}{\partial x_i} \quad \text{and} \quad \mathrm{curl}\,(U) = \left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)_{1\leq i,j\leq d}.$$
- For a matrix-valued function $\Sigma : \Omega \longrightarrow \mathbb{R}^{d\times d}$,
$$\mathrm{Div}\,(\Sigma) = \left(\sum_{i=1}^{d}\frac{\partial \Sigma_{i,j}}{\partial x_i}\right)_{1\leq j\leq d} \quad \text{and} \quad \mathrm{Curl}\,(\Sigma) = \left(\frac{\partial \Sigma_{i,k}}{\partial x_j} - \frac{\partial \Sigma_{j,k}}{\partial x_i}\right)_{1\leq i,j,k\leq d}.$$
- If H is a vector space endowed with a norm $\|\cdot\|$, the equality $g_\varepsilon(h) = o_H(|h|^n)$, for $n \in \mathbb{N}$, means that
$$\lim_{h\to 0}\left(\frac{1}{|h|^n}\sup_{\varepsilon>0}\|g_\varepsilon(h)\|\right) = 0. \tag{1.13}$$

- For $k \in \mathbb{N}$, $\mathscr{C}_c^k(\Omega)$ denotes the space of k-continuously derivable functions with compact support in Ω.

- For $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$, we denote

$$|\nu| = \nu_1 + \cdots + \nu_d \quad \text{and} \quad \frac{\partial^{|\nu|}}{\partial x^\nu} = \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_d}}{\partial x_d^{\nu_d}}.$$

- $Y := (0,1)^3$, and the Y-average is denoted $\langle \cdot \rangle$.

- $H_\sharp(Y; Z)$ denotes the space of the Y-periodic functions from $\mathbb{R}^3$ to Z which belong to $H_{\rm loc}(\mathbb{R}^d)$ for a generic function space H.

Remark 1.1. Consider a sequence $g_\varepsilon(h)$ in H which satisfies the expansion of order $n \in \mathbb{N}$,

$$g_\varepsilon(h) = g_\varepsilon^0 + g_\varepsilon^1(h) + \cdots + g_\varepsilon^n(h, \dots, h) + o_H(|h|^n), \tag{1.14}$$

where for any $k \leq n$, $h \mapsto g_\varepsilon^k(h, \dots, h)$ are bounded sequences of k-linear symmetric forms in H. In view of (1.13) each term $g_\varepsilon^k(h, \dots, h)$ of the expansion inherits of the same (weak or strong) convergence of $g_\varepsilon(h)$ in H.

Let us recall the definition of the H-convergence due to Murat, Tartar [18]:

Definition 1.1 (Murat, Tartar [18])**.** *A sequence A_ε in $\mathcal{M}(\alpha, \beta; \Omega)$ is said to H-converge to the matrix-valued function A_* if for any distribution $f \in H^{-1}(\Omega)$, the solution $u_\varepsilon \in H_0^1(\Omega)$ of the equation* $\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f$ *satisfies the convergences*

$$u_\varepsilon \rightharpoonup u_* \text{ weakly in } H_0^1(\Omega) \quad \text{and} \quad A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_* \nabla u_* \text{ weakly in } L^2(\Omega)^2, \tag{1.15}$$

where u_* solves in $H_0^1(\Omega)$ the homogenized equation $\operatorname{div}(A_* \nabla u_*) = f$.

Murat and Tartar [18] proved that for any sequence A_ε in $\mathcal{M}(\alpha, \beta; \Omega)$, there exist A_* in $\mathcal{M}(\alpha, \beta; \Omega)$ and a subsequence of A_ε which H-converges to A_*.

2 The three-dimensional effective magneto-resistance

2.1 The three-dimensional Hall effect and magneto-resistance

Let $\alpha, \beta > 0$, and let Ω be a regular bounded domain of $\mathbb{R}^3$. Consider a heterogeneous conductor in Ω, with a symmetric matrix-valued conductivity $\sigma_\varepsilon \in \mathcal{M}(\alpha, \beta; \Omega)$ (see (1.12)), associated with the resistivity $\rho_\varepsilon := (\sigma_\varepsilon)^{-1}$. Here, ε is a small positive parameter which represents the scale of the microstructure. In the presence of a magnetic field $h \in \mathbb{R}^3$, it is known (see, *e.g.*, [14]) that the perturbed resistivity satisfies the property

$$\rho_\varepsilon(-h) = \rho_\varepsilon(h)^{\rm T}. \tag{2.1}$$

Also assume that the conductivity satisfies the following regularity properties: there exist an open ball O in $\mathbb{R}^3$ centered at 0 and $b \in L^\infty(\Omega)$ such that for any $\varepsilon > 0$ and any multi-index $|\nu| \leq 2$,

$$\begin{cases} \sigma_\varepsilon(h) \in \mathcal{M}(\alpha, \beta; \Omega), \quad \forall\, h \in O, \\ h \mapsto \sigma_\varepsilon(h)(x) \text{ is of class } \mathscr{C}^{|\nu|} \text{ on } O, \quad \forall\, x \in \Omega, \\ \left| \dfrac{\partial^{|\nu|}\sigma_\varepsilon}{\partial h^\nu}(h)(x) - \dfrac{\partial^{|\nu|}\sigma_\varepsilon}{\partial h^\nu}(k)(x) \right| \leq b(x)\ |h - k|, \quad \forall\, h, k \in O, \text{ a.e. } x \in \Omega. \end{cases} \tag{2.2}$$

As a consequence of (2.2), the resistivity $\rho_\varepsilon(h)$ satisfies the second-order expansion

$$\rho_\varepsilon(h) = \rho_\varepsilon + \mathscr{R}_\varepsilon(h) + \mathscr{M}_\varepsilon(h,h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^2), \tag{2.3}$$

where $\mathscr{R}_\varepsilon : \mathbb{R}^3 \to L^\infty(\Omega)^{3\times 3}$ and $\mathscr{M}_\varepsilon : \mathbb{R}^3 \times \mathbb{R}^3 \to L^\infty(\Omega)^{3\times 3}$ are sequences of linear operators uniformly bounded with respect to ε. By virtue of (2.1), for any h in $\mathbb{R}^3$, $\rho_\varepsilon(0)$ and $\mathscr{M}_\varepsilon(h,h)$ are symmetric matrix-valued functions, while $\mathscr{R}_\varepsilon(h)$ is an antisymmetric matrix-valued function. The matrix-valued function defined by (see (1.9) and [8] for more details)

$$R_\varepsilon h := \mathscr{E}^{-1}\big(\mathscr{R}_\varepsilon(h)\big), \tag{2.4}$$

and the second-order term $\mathscr{M}_\varepsilon(h,h)$ are respectively called the Hall matrix and the magneto-resistance associated with the perturbed resistivity $\rho_\varepsilon(h)$, so that

$$\rho_\varepsilon(h) = \rho_\varepsilon + \mathscr{E}(R_\varepsilon h) + \mathscr{M}_\varepsilon(h,h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^2). \tag{2.5}$$

Remark 2.1. Since by assumption $\mathscr{R}_\varepsilon : \mathbb{R}^3 \to L^\infty(\Omega)^{3\times 3}$ is uniformly bounded with respect to ε, R_ε is a bounded sequence in $L^\infty(\Omega)^{3\times 3}$.

Similarly, we define the linear operators $S_\varepsilon : \mathbb{R}^3 \to L^\infty(\Omega)^{3\times 3}$ and $\mathscr{N}_\varepsilon : \mathbb{R}^3 \times \mathbb{R}^3 \to L^\infty(\Omega)^{3\times 3}$ by

$$\sigma_\varepsilon(h) = \rho_\varepsilon(h)^{-1} = \sigma_\varepsilon + \mathscr{E}(S_\varepsilon h) + \mathscr{N}_\varepsilon(h,h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^2), \tag{2.6}$$

which are uniformly bounded with respect to ε.

2.2 Homogenization of the magneto-resistance

Assume that $\sigma_\varepsilon(h)$ H-converges (see definition 1.1) to $\sigma_*(h)$ for any $h \in O$. In fact due to the compactness of H-convergence [18] this holds true for a subsequence of ε and a countable set of h. Then, by [9] (Theorem 2.5 in the symmetric case) and [3] (Theorem 3.1 in the non symmetric case), the effective (or homogenized) conductivity $\sigma_*(h)$ satisfies $\sigma_*(-h) = \sigma_*(h)^{\mathrm{T}}$. By the regularity conditions (2.2) (see [5] and [9] for more details), as in (2.6), we have the second-order expansion

$$\sigma_*(h) = \sigma_* + \mathscr{E}(S_* h) + \mathscr{N}_*(h,h) + o(|h|^2). \tag{2.7}$$

Moreover, by taking the inverse of (2.7), the effective resistivity $\rho_*(h) := \sigma_*(h)^{-1}$ also expands as

$$\rho_*(h) = \rho_* + \mathscr{E}(R_* h) + \mathscr{M}_*(h,h) + o(|h|^2), \tag{2.8}$$

where $R_* \in L^\infty(\Omega)^{3\times 3}$ is the effective Hall matrix and $\mathscr{M}_* : \mathbb{R}^3 \times \mathbb{R}^3 \to L^\infty(\Omega)^{3\times 3}$ is the effective magneto-resistance tensor of the composite. We have the following result:

Proposition 2.1. *The following relations hold for any $h \in O$,*

$$S_\varepsilon = -\,\mathrm{Cof}(\sigma_\varepsilon)\,R_\varepsilon \quad \textit{and} \quad S_* = -\,\mathrm{Cof}(\sigma_*)\,R_*, \tag{2.9}$$

$$\mathscr{N}_\varepsilon(h,h) = -\,\sigma_\varepsilon\,\mathscr{M}_\varepsilon(h,h)\,\sigma_\varepsilon + \mathscr{E}(S_\varepsilon h)\,\sigma_\varepsilon^{-1}\,\mathscr{E}(S_\varepsilon h), \tag{2.10}$$

$$\mathscr{N}_*(h,h) = -\,\sigma_*\,\mathscr{M}_*(h,h)\,\sigma_* + \mathscr{E}(S_* h)\,\sigma_*^{-1}\,\mathscr{E}(S_* h). \tag{2.11}$$

Proof. By the first-order expansions (2.5) and (2.6) we have, for any $h \in O$ and any $t > 0$ small enough,

$$\begin{aligned}0 &= -I_3 + \big(\sigma_\varepsilon + \mathscr{E}(tS_\varepsilon h) + \mathscr{N}_\varepsilon(th,th)\big)\big(\rho_\varepsilon + \mathscr{E}(tR_\varepsilon h) + \mathscr{M}_\varepsilon(th,th)\big) + o_{L^\infty(\Omega)^{3\times3}}(t^2|h|^2)\\ &= t\big(\mathscr{E}(S_\varepsilon h)\sigma_\varepsilon^{-1} + \sigma_\varepsilon\mathscr{E}(R_\varepsilon h)\big)\\ &\quad + t^2\big(\mathscr{N}_\varepsilon(h,h)\sigma_\varepsilon^{-1} + \mathscr{E}(S_\varepsilon h)\mathscr{E}(R_\varepsilon h) + \sigma_\varepsilon\mathscr{M}_\varepsilon(h,h)\big) + o_{L^\infty(\Omega)^{3\times3}}(t^2|h|^2).\end{aligned}$$

Then, dividing by t the previous equality and letting t tend to zero we obtain

$$\mathscr{E}(R_\varepsilon h) = -\,\sigma_\varepsilon^{-1}\,\mathscr{E}(S_\varepsilon h)\,\sigma_\varepsilon^{-1}. \tag{2.12}$$

Hence, we get that

$$0 = t^2\big(\mathscr{N}_\varepsilon(h,h)\;\sigma_\varepsilon^{-1} - \mathscr{E}(S_\varepsilon h)\;\sigma_\varepsilon^{-1}\;\mathscr{E}(S_\varepsilon h)\;\sigma_\varepsilon^{-1} + \sigma_\varepsilon\;\mathscr{M}_\varepsilon(h,h)\big) + o_{L^\infty(\Omega)^{3\times3}}(t^2|h|^2). \tag{2.13}$$

We divide this equality by t^2 and let t tend to 0 to get (2.10).

The first equality of (2.9) is a straightforward consequence of the following algebraic lemma which is proved in [8] (Lemma 1):

$$\forall\, P \in \mathbb{R}^{3\times3},\quad P^{\mathrm{T}}\mathscr{E}(\xi)P = \mathscr{E}\big(\mathrm{Cof}(P)^{\mathrm{T}}\xi\big). \tag{2.14}$$

Applying (2.14) to the equality (2.30), and using that σ_ε is symmetric, we get that for any $h \in O$,

$$\mathscr{E}(S_\varepsilon h) = -\sigma_\varepsilon\mathscr{E}(R_\varepsilon h)\sigma_\varepsilon = -\sigma_\varepsilon^{\mathrm{T}}\mathscr{E}(R_\varepsilon h)\sigma_\varepsilon = \mathscr{E}\big(-\mathrm{Cof}(\sigma_\varepsilon)^{\mathrm{T}}R_\varepsilon h\big) = \mathscr{E}\big(-\mathrm{Cof}(\sigma_\varepsilon)R_\varepsilon h\big), \tag{2.15}$$

which shows the first part of (2.9) due to the invertibility of $\mathscr{E}$.

The proof for the homogenized quantities in (2.9) and (2.11) is quite similar. □

2.2.1 The general case

We now give an analogous in dimension three of Theorem 2.2 of [4] in order to give weak convergences of the effective Hall matrix and the magneto-resistance. All the subsequences parametrized by h converge up to a subsequence of ε. Due to (2.2), the linearity or the quadratic dependence in h, the convergences hold for any h. From now on, we consider a subsequence still denoted by ε such that all the sequences converge as ε tends to 0 and for any h in O.

First of all, we need to introduce a corrector $P_\varepsilon(h)$ (or electric field) in the sense of Murat-Tartar (see [18]), which is the gradient of a vector-valued $U_\varepsilon(h)$ associated with the unperturbed conductivity σ_ε in $\mathcal{M}(\alpha,\beta;\Omega)$. To this end consider the solution $U_\varepsilon(h)$ in $H^1(\Omega)^3$ of the problem

$$\begin{cases}\mathrm{Div}\big(\sigma_\varepsilon(h)DU_\varepsilon(h)\big) &=& \mathrm{Div}\big(\sigma_*(h)\big) & \text{in } \mathscr{D}'(\Omega),\\ U_\varepsilon(h)(x) - x &=& 0 & \text{on } \partial\Omega.\end{cases} \tag{2.16}$$

Thanks to H-convergence and the Meyers estimate of [16], there exists a number $p > 2$ which only depends on α, β, Ω, such that the corrector $P_\varepsilon(h) := DU_\varepsilon(h)$ satisfies, for any $h \in O$,

$$P_\varepsilon(h) \rightharpoonup I_3 \quad \text{weakly in } L^p(\Omega)^{3\times3}. \tag{2.17}$$

The knowledge of such a corrector combined with the div-curl lemma (see [17] and [20]) permits to derive the effective perturbed effective conductivity by the following convergence

$$\sigma_\varepsilon(h)P_\varepsilon(h) \rightharpoonup \sigma_*(h) \quad \text{weakly in } L^p(\Omega)^{3\times3}. \tag{2.18}$$

By the regularity condition (2.2), the coercivity of $\sigma_\varepsilon(h)$ and the Meyers estimate [16], the potential $U_\varepsilon(h)$ and the corrector $P_\varepsilon(h)$ admit the following second-order expansions in h:

$$U_\varepsilon(h) = U_\varepsilon^0 + U_\varepsilon^1(h) + U_\varepsilon^2(h,h) + o_{W^{1,p}(\Omega)^3}(|h|^2), \tag{2.19}$$

$$P_\varepsilon(h) = P_\varepsilon^0 + P_\varepsilon^1(h) + P_\varepsilon^2(h,h) + o_{L^p(\Omega)^{3\times3}}(|h|^2). \tag{2.20}$$

We can now state the following result:

Theorem 2.1. *Assume that the conditions (2.1)-(2.8) are satisfied, and that the norms of the Hall matrix R_ε and the local magneto-resistance tensor $\mathscr{M}_\varepsilon$ are bounded in $L^\infty(\Omega)$. Then, the effective Hall matrix R_*, the effective S-matrix S_* and the effective magneto-resistance are given by the following limits, for any $h \in O$,*

$$S_* = \lim_{w-L^1(\Omega)} \mathrm{Cof}(P_\varepsilon^0)^T S_\varepsilon, \quad \mathrm{Cof}(\sigma_*)\, R_* = \lim_{w-L^1(\Omega)} \mathrm{Cof}(\sigma_\varepsilon P_\varepsilon^0)^T R_\varepsilon, \tag{2.21}$$

and

$$\begin{aligned}\sigma_* \mathscr{M}_*(h,h)\,\sigma_* &= \lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon P_\varepsilon^0\big)^T \mathscr{M}_\varepsilon(h,h)\,\big(\sigma_\varepsilon P_\varepsilon^0\big) - \mathscr{E}(S_* h)^T \sigma_*^{-1}\, \mathscr{E}(S_* h) \\ &+ \lim_{w-L^1(\Omega)} \big(\mathscr{E}(S_\varepsilon h) P_\varepsilon^0 + \sigma_\varepsilon P_\varepsilon^1(h)\big)^T \sigma_\varepsilon^{-1} \big(\mathscr{E}(S_\varepsilon h) P_\varepsilon^0 + \sigma_\varepsilon P_\varepsilon^1(h)\big),\end{aligned} \tag{2.22}$$

where $w - L^1(\Omega)$ means that the convergence holds weakly in $L^1(\Omega)$ and P_ε^0, $P_\varepsilon^1(h)$ are the matrix-valued gradient which satisfy (2.20).

Proof. The proof uses similar expansions as in [5] combined with algebraic specificities of dimension 3. Taking into account the expansions (2.6) and (2.20), we have:

$$\begin{aligned}\sigma_\varepsilon(h) P_\varepsilon(h) &= \sigma_\varepsilon P_\varepsilon^0 + \big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big) \\ &+ \big(\sigma_\varepsilon P_\varepsilon^2(h,h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^1(h) + \mathscr{N}_\varepsilon(h,h) P_\varepsilon^0\big) + o_{L^2(\Omega)^{3\times 3}}(|h|^2).\end{aligned} \tag{2.23}$$

By virtue of Remark 1.1, using the properties (2.16)-(2.18) satisfied by the corrector $P_\varepsilon(h)$ in the expansions (2.20), (2.23) and (2.7), we get that

$$\begin{cases} P_\varepsilon^0 \rightharpoonup I_3 & \text{weakly in } L^p(\Omega)^{3\times 3}, \\ P_\varepsilon^1(h) \rightharpoonup 0 & \text{weakly in } L^p(\Omega)^{3\times 3}, \\ P_\varepsilon^2(h,h) \rightharpoonup 0 & \text{weakly in } L^p(\Omega)^{3\times 3}, \end{cases} \tag{2.24}$$

and

$$\begin{cases} \mathrm{Div}\big(\sigma_\varepsilon P_\varepsilon^0\big) = \mathrm{Div}\big(\sigma_*\big), \\ \mathrm{Div}\big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big) = \mathrm{Div}\big(\mathscr{E}(S_* h)\big) \end{cases} \quad \text{in } \mathscr{D}'(\Omega)^{3\times 3}. \tag{2.25}$$

Moreover, from the expansions (2.20), (2.23) and the symmetry of σ_ε, we deduce that

$$\begin{aligned} P_\varepsilon(h)^{\mathrm T} \sigma_\varepsilon(h) P_\varepsilon(h) &= (P_\varepsilon^0)^{\mathrm T} \sigma_\varepsilon P_\varepsilon^0 + (P_\varepsilon^0)^{\mathrm T} \mathscr{E}(S_\varepsilon h) P_\varepsilon^0 + (P_\varepsilon^0)^{\mathrm T} \big(\mathscr{E}(S_\varepsilon h) P_\varepsilon^1(h) + \mathscr{N}_\varepsilon(h,h) P_\varepsilon^0\big) \\ &+ (\sigma_\varepsilon P_\varepsilon^0)^{\mathrm T} P_\varepsilon^2(h,h) + (\sigma_\varepsilon P_\varepsilon^0)^{\mathrm T} P_\varepsilon^1(h) \\ &+ \big(P_\varepsilon^1(h)\big)^{\mathrm T} \sigma_\varepsilon(h) P_\varepsilon(h) + \big(P_\varepsilon^2(h,h)\big)^{\mathrm T} \sigma_\varepsilon(h) P_\varepsilon(h) + o_{L^{p/2}(\Omega)^{3\times 3}}(|h|^2). \end{aligned} \tag{2.26}$$

Then, taking into account (2.17), (2.24), (2.25), the div-curl lemma implies that $P_\varepsilon(h)^{\mathrm T} \sigma_\varepsilon(h) P_\varepsilon(h)$ converges to $\sigma_*(h)$ in $L^{p/2}(\Omega)^{3\times 3}$, and $\big(P_\varepsilon^1(h)\big)^{\mathrm T} \sigma_\varepsilon(h) P_\varepsilon(h)$, $\big(P_\varepsilon^2(h,h)\big)^{\mathrm T} \sigma_\varepsilon(h) P_\varepsilon(h)$, $(\sigma_\varepsilon P_\varepsilon^0)^{\mathrm T} P_\varepsilon^2(h,h)$, $(\sigma_\varepsilon P_\varepsilon^0)^{\mathrm T} P_\varepsilon^1(h)$ converges to 0 in $L^{p/2}(\Omega)^{3\times 3}$. Noting that $\big(P_\varepsilon^0\big)^{\mathrm T} \mathscr{E}(S_\varepsilon h) P_\varepsilon^0 = \mathscr{E}\big(\mathrm{Cof}(P_\varepsilon^0)^{\mathrm T} S_\varepsilon h\big)$ by (2.14) and passing to the limit in (2.26), we obtain

$$\begin{aligned} \sigma_*(h) &= \sigma_* + \lim_{w-L^1(\Omega)} \mathscr{E}\big(\mathrm{Cof}(P_\varepsilon^0)^{\mathrm T} S_\varepsilon h\big) \\ &+ \lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm T} \mathscr{E}(S_\varepsilon h) P_\varepsilon^1(h) + \big(P_\varepsilon^0\big)^{\mathrm T} \mathscr{N}_\varepsilon(h,h) P_\varepsilon^0\Big] + o_{L^1(\Omega)^{3\times 3}}(|h|^2). \end{aligned} \tag{2.27}$$

Equating this expression with (2.7) it follows that

$$\mathscr{E}\big(S_* h\big) = \lim_{w-L^1(\Omega)} \mathscr{E}\big(\mathrm{Cof}(P_\varepsilon^0)^{\mathrm T} S_\varepsilon h\big), \tag{2.28}$$

and

$$\mathscr{N}_*(h,h) = \lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{E}(S_\varepsilon h) P_\varepsilon^1(h) + \big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{N}_\varepsilon(h,h) P_\varepsilon^0\Big]. \tag{2.29}$$

As $\mathscr{E}$ is an invertible linear mapping, we deduce from (2.28) and (2.9) the convergences (2.21).

We have, as $\mathscr{E}(S_\varepsilon h)$ is antisymmetric and σ_ε symmetric,

$$\begin{aligned}
&\big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big)^{\mathrm{T}} \big(\sigma_\varepsilon\big)^{-1} \big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big) \\
&= -\big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{E}(S_\varepsilon h) \big(\sigma_\varepsilon\big)^{-1} \mathscr{E}(S_\varepsilon h) P_\varepsilon^0 - \big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{E}(S_\varepsilon h) P_\varepsilon^1(h) + \big(P_\varepsilon^1(h)\big)^{\mathrm{T}} \big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big).
\end{aligned} \tag{2.30}$$

Again, taking into account (2.17), (2.24), (2.25), the div-curl lemma implies that

$$\big(P_\varepsilon^1(h)\big)^{\mathrm{T}} \big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big) \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega)^{3\times 3}. \tag{2.31}$$

Hence, (2.30) implies that

$$\begin{aligned}
&\lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^{-1} \big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big) \\
&= -\lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{E}(S_\varepsilon h) \sigma_\varepsilon^{-1} \mathscr{E}(S_\varepsilon h) P_\varepsilon^0 + \big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{E}(S_\varepsilon h) P_\varepsilon^1(h)\Big].
\end{aligned} \tag{2.32}$$

Combining the equalities (2.10), (2.11) of Proposition 2.1 with (2.29), (2.32) and the antisymmetry of $\mathscr{E}(S_\varepsilon h)$ and $\mathscr{E}(S_* h)$, we obtain that

$$\begin{aligned}
&\sigma_* \,\mathscr{M}_*(h,h)\, \sigma_* - \lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{M}_\varepsilon(h,h) \big(\sigma_\varepsilon P_\varepsilon^0\big) \\
&= \lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{N}_\varepsilon(h,h) P_\varepsilon^0 - \big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{E}(S_\varepsilon h) \sigma_\varepsilon^{-1} \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\Big] - \mathscr{N}_*(h,h) + \mathscr{E}(S_* h)\ \sigma_*^{-1} \mathscr{E}(S_* h) \\
&= -\lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{E}(S_\varepsilon h) \sigma_\varepsilon^{-1} \mathscr{E}(S_\varepsilon h) P_\varepsilon^0 + \big(P_\varepsilon^0\big)^{\mathrm{T}} \mathscr{E}(S_\varepsilon h) P_\varepsilon^1(h)\Big] - \mathscr{E}(S_* h)^{\mathrm{T}} \sigma_*^{-1} \mathscr{E}(S_* h) \\
&= \lim_{w-L^1(\Omega)} \Big[\big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big)^{\mathrm{T}} \big(\sigma_\varepsilon\big)^{-1} \big(\sigma_\varepsilon P_\varepsilon^1(h) + \mathscr{E}(S_\varepsilon h) P_\varepsilon^0\big)\Big] - \mathscr{E}(S_* h)^{\mathrm{T}} \sigma_*^{-1} \mathscr{E}(S_* h).
\end{aligned}$$

In fact, the convergences (2.21) and (2.22) hold in $L^{p/2}(\Omega)^{3\times 3}$. □

2.2.2 The periodic case

We now give a corollary of Theorem 2.1 for periodic media. To this end, set $Y := (0,1)^3$, and consider the εY-periodic conductivity

$$\sigma_\varepsilon(h)(x) := \sigma(h)\left(\frac{x}{\varepsilon}\right), \tag{2.33}$$

where $\sigma(h)$ is a Y-periodic matrix-valued function. We assume (2.1) and analogous regularity conditions to (2.2): there exists an open ball O in $\mathbb{R}^3$ centered at 0 such that

$$\begin{cases} \sigma(h) \in \mathcal{M}(\alpha,\beta;\Omega), \quad \forall\, h \in O, \\ h \mapsto \sigma(h)(y) \text{ is of class } \mathscr{C}^2 \text{ on } O, \quad \forall\, y \in Y. \end{cases} \tag{2.34}$$

These conditions gives the expansions, like in (2.6) and (2.3)

$$\sigma(h) = \sigma + \mathscr{E}(Sh) + \mathscr{N}(h,h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^2), \tag{2.35}$$

$$\rho(h) = \sigma(h)^{-1} = \rho + \mathscr{E}(Rh) + \mathscr{M}(h,h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^2), \tag{2.36}$$

where σ, ρ, S, R, $\mathscr{N}$ and $\mathscr{M}$ are Y-periodic functions bounded in Y. The corrector $P_\varepsilon(h) := DU_\varepsilon(h)$ defined in (2.16) and (2.19) reads as $U_\varepsilon(h) := \varepsilon U(h)\left(\frac{x}{\varepsilon}\right)$, where $U(h)$ is the unique solution in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ (up to an additive constant) of the problem

$$\begin{cases} \mathrm{Div}\big(\sigma(h) DU(h)\big) &= 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^3), \\ y \mapsto U(h)(y) - y &= 0 \quad \text{is } Y\text{-periodic}. \end{cases} \tag{2.37}$$

and $P(h) := DU(h)$ with

$$\langle P(h)\rangle = I_3. \tag{2.38}$$

We have the classical periodic homogenization formula (see, *e.g.*, [18] for more details) $\sigma_*(h) = \langle \sigma(h)DU(h)\rangle$. By virtue of (2.34) we have expansions similar to (2.7), (2.8) and (2.20)

$$\sigma_*(h) = \sigma_* + \mathscr{E}(S_*h) + \mathscr{N}_*(h,h) + o(|h|^2), \tag{2.39}$$

$$\rho_*(h) = \sigma_*(h)^{-1} = \rho_* + \mathscr{E}(R_*h) + \mathscr{M}_*(h,h) + o(|h|^2), \tag{2.40}$$

and

$$P(h) = P^0 + P^1(h) + P^2(h,h) + o_{L^2(\Omega)^{3\times 3}}(|h|^2). \tag{2.41}$$

We can state a corollary to Theorem 2.1:

Corollary 2.1. *For a periodic conductor, the effective Hall matrix R_*, the effective S-matrix S_* and the effective magneto-resistance tensor $\mathscr{M}_*$ are given by the following relations, for any $h \in O$,*

$$S_* = \big\langle \mathrm{Cof}(P^0)^T S\big\rangle, \quad \mathrm{Cof}(\sigma_*)\ R_* = \big\langle \mathrm{Cof}(\sigma_\varepsilon P^0)^T R\big\rangle, \tag{2.42}$$

and

$$\begin{aligned}\sigma_*\,\mathscr{M}_*(h,h)\,\sigma_* \;&=\; \Big\langle \big(\sigma P^0\big)^T \mathscr{M}(h,h)\big(\sigma P^0\big)\Big\rangle - \mathscr{E}(S_*h)^T\sigma_*^{-1}\mathscr{E}(S_*h)\\ &+\; \Big\langle \big(\mathscr{E}(Sh)P^0 + \sigma P^1(h)\big)^T\sigma^{-1}\big(\mathscr{E}(Sh)P^0 + \sigma P^1(h)\big)\Big\rangle,\end{aligned} \tag{2.43}$$

where P^0, $P^1(h)$ are the matrix-valued gradient which satisfy (2.41).

3 Comparison between the effective magneto-resistance and the local magneto-resistance.

3.1 The main result

We now give a generalization of the two-dimensional Theorem 2.4 of [4], and a corollary in the periodic case with the notations of Section 2.2.2.

Theorem 3.1. *Assume that the conditions (2.1)-(2.8) are satisfied, and that the norm of the local Hall matrix R_ε and the norm of the local magneto-resistance tensor $\mathscr{M}_\varepsilon$ are bounded in $L^\infty(\Omega)$. Then, for any $h \in O$, we have*

$$\sigma_*\,\mathscr{M}_*(h,h)\,\sigma_* \ge \lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon P^0_\varepsilon\big)^T \mathscr{M}_\varepsilon(h,h)\big(\sigma_\varepsilon P^0_\varepsilon\big). \tag{3.1}$$

Moreover, (3.1) is an equality if and only if

$$\mathrm{Curl}\,\big(\mathscr{E}(R_\varepsilon h)\sigma_\varepsilon P^0_\varepsilon\big) \ \textit{ lies in a compact subset of } H^{-1}(\Omega)^{3\times 3\times 3}. \tag{3.2}$$

Corollary 3.1. *In the periodic case, the constant effective magneto-resistance tensor $\mathscr{M}_*$ and the constant effective conductivity σ_* satisfy the inequality for any $h \in O$,*

$$\sigma_*\,\mathscr{M}_*(h,h)\,\sigma_* \ge \Big\langle \big(\sigma P^0\big)^T \mathscr{M}(h,h)\big(\sigma P^0\big)\Big\rangle, \quad \textit{with } \sigma_* = \big\langle \sigma P^0\big\rangle, \tag{3.3}$$

where $\sigma(y)$ is the local conductivity and $\mathscr{M}(h,h)(y)$ is the local magneto-resistance. Moreover, (3.3) is an equality if and only if

$$\mathrm{Curl}\,\big(\mathscr{E}(Rh)\,\sigma\,P^0\big) = 0 \quad \textit{in } \mathscr{D}'(\mathbb{R}^3)^{3\times 3\times 3}. \tag{3.4}$$

Remark 3.1. Then the inequality (3.3) can be written, for any $h \in \mathbb{R}^3$,

$$\mathscr{M}_*(h,h)\langle j\rangle\cdot\langle j\rangle \geq \big\langle \mathscr{M}(h,h)j\cdot j\big\rangle, \quad \text{with } \langle j\rangle = \sigma_*\langle e\rangle, \tag{3.5}$$

where $e(y) = P^0(y)\langle e\rangle$ is the local electric field and $j(y) = \sigma(y)e(y)$ is the local current field. Inequality (3.5) means that the dissipation energy in a composite is greater than or equal to the average of the dissipation energy in each of its phases.

The proof of Theorem 3.1 is based on the following result:

Lemma 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider a sequence A_ε of symmetric matrix-valued functions in $\mathcal{M}(\alpha,\beta;\Omega)$ which H-converges to A_*, and a sequence ξ_ε of $L^2(\Omega)^d$ which satisfies*

$$\xi_\varepsilon \rightharpoonup \xi \text{ weakly in } L^2(\Omega)^d \quad \text{and} \quad \operatorname{div}(\xi_\varepsilon) \longrightarrow \operatorname{div}(\xi) \text{ strongly in } H^{-1}(\Omega). \tag{3.6}$$

Also assume that

$$A_\varepsilon^{-1}\xi_\varepsilon\cdot\xi_\varepsilon \rightharpoonup \zeta \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{3.7}$$

Then, we have the inequality

$$\zeta \geq A_*^{-1}\xi\cdot\xi \quad \text{in } \mathcal{M}(\Omega). \tag{3.8}$$

Moreover, the inequality (3.8) is an equality if and only if

$$\operatorname{curl}\big(A_\varepsilon^{-1}\xi_\varepsilon\big) \text{ lies in a compact subset of } H^{-1}_{\mathrm{loc}}(\Omega)^{d\times d}. \tag{3.9}$$

Remark 3.2. Inequality (3.8) is a classical duality inequality in the periodic case (see [12] pp.160–200). However, up our knowledge the non-periodic case and the condition (3.9) of equality are less classical and deserve a proof.

Proof of Theorem 3.1. Taking into account the expansions (2.6) and (2.20), we have as in (2.23):

$$\sigma_\varepsilon(h)P_\varepsilon(h) \quad = \quad \sigma_\varepsilon P^0_\varepsilon + \big(\sigma_\varepsilon P^1_\varepsilon(h) + \mathscr{E}(S_\varepsilon h)P^0_\varepsilon\big) + o_{L^p(\Omega)^{3\times 3}}(h). \tag{3.10}$$

By virtue of Remark 1.1, using the properties (2.16)-(2.18) satisfied by the corrector $P_\varepsilon(h)$ in the expansions (2.20), (2.23) and (2.7), we have, like in (2.24) and (2.25):

$$\begin{cases} P^0_\varepsilon & \rightharpoonup & I_3 & \text{weakly in } L^p(\Omega)^{3\times 3}, \\ P^1_\varepsilon(h) & \rightharpoonup & 0 & \text{weakly in } L^p(\Omega)^{3\times 3}, \\ \sigma_\varepsilon P^1_\varepsilon(h) + \mathscr{E}(S_\varepsilon h)P^0_\varepsilon & \rightharpoonup & \mathscr{E}(S_* h) & \text{weakly in } L^p(\Omega)^{3\times 3}. \end{cases} \tag{3.11}$$

Let $\lambda \in \mathbb{R}^3$. We apply Lemma 3.1 with $A_\varepsilon := \sigma_\varepsilon$, $\xi_\varepsilon := \big(\sigma_\varepsilon P^1_\varepsilon(h) + \mathscr{E}(S_\varepsilon h)P^0_\varepsilon\big)\lambda$, $\xi := \mathscr{E}(S_* h)\lambda$ and

$$\zeta := \lim_{w-L^1(\Omega)} \big(\mathscr{E}(S_\varepsilon h)P^0_\varepsilon + \sigma_\varepsilon P^1_\varepsilon(h)\big)^{\mathrm{T}}\sigma_\varepsilon^{-1}\big(\mathscr{E}(S_\varepsilon h)P^0_\varepsilon + \sigma_\varepsilon P^1_\varepsilon(h)\big)\lambda\cdot\lambda. \tag{3.12}$$

It follows that for any $\lambda \in \mathbb{R}^3$,

$$\lim_{w-L^1(\Omega)} \big(\mathscr{E}(S_\varepsilon h)P^0_\varepsilon + \sigma_\varepsilon P^1_\varepsilon(h)\big)^{\mathrm{T}}\sigma_\varepsilon^{-1}\big(\mathscr{E}(S_\varepsilon h)P^0_\varepsilon + \sigma_\varepsilon P^1_\varepsilon(h)\big)\lambda\cdot\lambda \geq \mathscr{E}(S_* h)^{\mathrm{T}}\sigma_*^{-1}\mathscr{E}(S_* h)\lambda\cdot\lambda. \tag{3.13}$$

Using the fact that $P^1_\varepsilon(h)$ is a gradient and (2.12), (3.13) is an equality if and only if

$$\operatorname{Curl}\big(A_\varepsilon^{-1}\xi_\varepsilon\big) = \operatorname{Curl}\big(\sigma_\varepsilon^{-1}\mathscr{E}(S_\varepsilon h)P^0_\varepsilon\lambda\big) = -\operatorname{Curl}\big(\mathscr{E}(R_\varepsilon h)\sigma_\varepsilon P^0_\varepsilon\lambda\big) \tag{3.14}$$

lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)^{3\times 3}$. Due to the arbitrariness of λ, this can be rewritten

$$\lim_{w-L^1(\Omega)} \big(\mathscr{E}(S_\varepsilon h)P^0_\varepsilon + \sigma_\varepsilon P^1_\varepsilon(h)\big)^{\mathrm{T}}\sigma_\varepsilon^{-1}\big(\mathscr{E}(S_\varepsilon h)P^0_\varepsilon + \sigma_\varepsilon P^1_\varepsilon(h)\big) \geq \mathscr{E}(S_* h)^{\mathrm{T}}\sigma_*^{-1}\mathscr{E}(S_* h), \tag{3.15}$$

which is an equality if and only if

$$\operatorname{Curl}\left(\mathscr{E}(R_\varepsilon h)\sigma_\varepsilon P_\varepsilon^0\right) \quad \text{lies in a compact subset of } H^{-1}_{\rm loc}(\Omega)^{3\times 3\times 3}. \tag{3.16}$$

We conclude to (3.1) by (2.22) and (3.15), and to (3.2) by (3.16). $\square$

Proof of Lemma 3.1.

Proof of inequality (3.8): Let φ be a non-negative function in $\mathscr{C}^0_c(\Omega)$. Let $\delta > 0$, and for $i = 1, \ldots, k$, let $\lambda_i \in \mathbb{R}^d$ and let ω_i be balls in Ω such that

$$\operatorname{supp}\varphi \subset \bigcup_{i=1}^k \omega_i \quad \text{and} \quad \sum_{i=1}^k \int_{\omega_i} |A_*^{-1}\xi - \lambda_i|^2 \,dx \le \delta. \tag{3.17}$$

We consider a partition of unity $(\psi_i)_{1\le i\le k}$ such that

$$\forall\, i = 1, \ldots, k, \quad \psi_i \in \mathscr{C}^\infty_c(\omega_i), \quad 1 \ge \psi_i \ge 0, \quad \sum_{i=1}^k \psi_i \equiv 1 \text{ in } \operatorname{supp}\varphi, \tag{3.18}$$

and a sequence of functions $(\widetilde{\psi}_i)_{1\le i\le k}$ such that

$$\forall\, i = 1, \ldots, k, \quad \widetilde{\psi}_i \in \mathscr{C}^\infty_c(\omega_i), \quad \widetilde{\psi}_i \equiv 1 \text{ in } \operatorname{supp}\psi_i. \tag{3.19}$$

For $i = 1, \ldots, k$, let v^i_ε be the unique solution of the problem

$$\begin{cases} \operatorname{div}\left(A_\varepsilon \nabla v^i_\varepsilon\right) &= \operatorname{div}\left(A_* \nabla(\widetilde{\psi}_i \lambda_i \cdot x)\right) \quad \text{in } \mathscr{D}'(\Omega) \\ \qquad v^i_\varepsilon &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

Thanks to the H-convergence of A_ε (see Definition 1.1) we have the convergence

$$\forall\, i = 1, \ldots, k, \quad \begin{cases} v^i_\varepsilon & \rightharpoonup & v^i = \widetilde{\psi}_i \lambda_i \cdot x & \text{weakly in } H^1_0(\Omega), \\ \nabla v^i & \equiv & \lambda_i & \text{in } \operatorname{supp}\psi_i. \end{cases} \tag{3.21}$$

More generally, for any $\lambda \in \mathbb{R}^d$, we consider the unique solution v^λ_ε of the problem

$$\begin{cases} \operatorname{div}\left(A_\varepsilon \nabla v_\varepsilon\right) &= \operatorname{div}\left(A_* \lambda\right) \quad \text{in } \mathscr{D}'(\Omega), \\ \qquad v_\varepsilon &= \lambda \cdot x \qquad \text{on } \partial\Omega. \end{cases} \tag{3.22}$$

Again, by the H-convergence of A_ε, we have the convergences

$$\begin{cases} v_\varepsilon & \rightharpoonup & \lambda \cdot x & \text{weakly in } H^1_0(\Omega), \\ A_\varepsilon \nabla v_\varepsilon & \rightharpoonup & A_*\lambda & \text{weakly in } L^2(\Omega)^d. \end{cases} \tag{3.23}$$

We have by (3.18)

$$\int_\Omega \zeta\varphi \,dx = \lim_{\varepsilon\to 0} \int_\Omega A_\varepsilon^{-1}\xi_\varepsilon \cdot \xi_\varepsilon \;\varphi \,dx = \sum_{i=1}^k \lim_{\varepsilon\to 0} \int_{\omega_i} A_\varepsilon^{-1}\xi_\varepsilon \cdot \xi_\varepsilon \;\psi_i\;\varphi \,dx. \tag{3.24}$$

Combining this inequality with, for $i = 1, \ldots, k$,

$$A_\varepsilon^{-1}\xi_\varepsilon \cdot \xi_\varepsilon - 2\xi_\varepsilon \cdot \nabla v^i_\varepsilon + A_\varepsilon \nabla v^i_\varepsilon \cdot \nabla v^i_\varepsilon = A_\varepsilon^{-1}\left(\xi_\varepsilon - A_\varepsilon \nabla v^i_\varepsilon\right) \cdot \left(\xi_\varepsilon - A_\varepsilon \nabla v^i_\varepsilon\right) \ge 0, \tag{3.25}$$

we obtain that

$$\int_\Omega \zeta\varphi \,dx \ge \sum_{i=1}^k \liminf_{\varepsilon\to 0} \int_{\omega_i} \left(2\xi_\varepsilon \cdot \nabla v^i_\varepsilon - A_\varepsilon \nabla v^i_\varepsilon \cdot \nabla v^i_\varepsilon\right)\psi_i\varphi \,dx. \tag{3.26}$$

By (3.20), (3.6) and (3.21), and by the classical div-curl lemma of [17, 18] we have

$$\xi_\varepsilon \cdot \nabla v_\varepsilon^i \rightharpoonup \xi \cdot \nabla v^i = \xi \cdot \lambda_i \quad \text{and} \quad A_\varepsilon \nabla v_\varepsilon^i \cdot \nabla v_\varepsilon^i \rightharpoonup A_* \nabla v^i \cdot \nabla v^i = A_* \lambda_i \cdot \lambda_i \tag{3.27}$$

weakly-$*$ in $\mathcal{M}(\omega_i)$. This combined with (3.26) and (3.18) yields

$$\begin{aligned} \int_\Omega \zeta \varphi \, \mathrm{d}x &\geq \sum_{i=1}^k \int_{\omega_i} \big(2\xi \cdot \lambda_i - A_* \lambda_i \cdot \lambda_i\big)\, \psi_i \, \varphi \, \mathrm{d}x \\ &= \sum_{i=1}^k \int_{\omega_i} A_*^{-1} \xi \cdot \xi \, \psi_i \, \varphi \, \mathrm{d}x - \sum_{i=1}^k \int_{\omega_i} A_* \big(A_*^{-1}\xi - \lambda_i\big) \cdot \big(A_*^{-1}\xi - \lambda_i\big)\, \psi_i \, \varphi \, \mathrm{d}x \\ &= \int_\Omega A_*^{-1} \xi \cdot \xi \, \varphi \, \mathrm{d}x - \sum_{i=1}^k \int_{\omega_i} A_* \big(A_*^{-1}\xi - \lambda_i\big) \cdot \big(A_*^{-1}\xi - \lambda_i\big)\, \psi_i \, \varphi \, \mathrm{d}x. \end{aligned} \tag{3.28}$$

Moreover, by the Cauchy-Schwarz inequality and $A_* \in \mathcal{M}(\alpha, \beta; \Omega)$, we have for $i = 1, \ldots, k$,

$$\int_{\omega_i} A_* \big(A_*^{-1}\xi - \lambda_i\big) \cdot \big(A_*^{-1}\xi - \lambda_i\big)\, \psi_i \, \varphi \, \mathrm{d}x \leq \beta \, \|\varphi\|_\infty \int_{\omega_i} |A_*^{-1}\xi - \lambda_i|^2 \, \mathrm{d}x. \tag{3.29}$$

Summing these inequalities on i together with (3.28) and (3.17), we finally get that

$$\int_\Omega \zeta \varphi \, \mathrm{d}x \geq \int_\Omega A_*^{-1} \xi \cdot \xi \varphi \, \mathrm{d}x - \beta \, \delta \, \|\varphi\|_\infty. \tag{3.30}$$

We conclude to (3.1) since δ is arbitrary.

Proof of the case of equality: Let us now prove that the equality in (3.8) implies (3.9). Consider a compact subset K of Ω, and a sequence $\Phi_\varepsilon \in H_0^1(\Omega)^{d\times d}$ such that

$$\Phi_\varepsilon \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega)^{d\times d} \quad \text{and} \quad \operatorname{supp} \Phi_\varepsilon \subset K. \tag{3.31}$$

By the definitions of the curl and the divergence, and by (3.31) we have

$$\big\langle \operatorname{curl}\big(A_\varepsilon^{-1}\xi_\varepsilon\big), \Phi_\varepsilon \big\rangle_{H^{-1}(\Omega)^{d\times d}, H_0^1(\Omega)^{d\times d}} = \int_K A_\varepsilon^{-1} \xi_\varepsilon \cdot \operatorname{Div}\big(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\big) \, \mathrm{d}x. \tag{3.32}$$

Consider a partition of unity $(\psi_i)_{1\leq i\leq k}$ such that

$$\forall \ i = 1, \ldots, k, \quad \psi_i \in \mathscr{C}_c^\infty(\omega_i), \quad 0 \leq \psi_i \leq 1, \quad \sum_{i=1}^k \psi_i \equiv 1 \text{ in } K, \tag{3.33}$$

the functions $\widetilde{\psi}_i$ defined by (3.19), and the function v_ε^i by (3.20). We decompose the equality (3.32) in two parts

$$\big\langle \operatorname{curl}\big(A_\varepsilon^{-1}\xi_\varepsilon\big), \Phi_\varepsilon \big\rangle_{H^{-1}(\Omega)^{d\times d}, H_0^1(\Omega)^{d\times d}} = I_\varepsilon + J_\varepsilon, \tag{3.34}$$

where

$$I_\varepsilon := \sum_{i=1}^k \int_{\omega_i} \big(A_\varepsilon^{-1}\xi_\varepsilon - \nabla v_\varepsilon^i\big) \cdot \operatorname{Div}\big(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\big)\, \psi_i \, \mathrm{d}x, \tag{3.35}$$

and

$$J_\varepsilon := \sum_{i=1}^k \int_{\omega_i} \nabla v_\varepsilon^i \cdot \operatorname{Div}\big(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\big)\, \psi_i \, \mathrm{d}x. \tag{3.36}$$

On the one hand, by the Cauchy-Schwarz inequality we have

$$|I_\varepsilon|^2 \leq \left(\sum_{i=1}^k \int_{\omega_i} |A_\varepsilon^{-1}\xi_\varepsilon - \nabla v_\varepsilon^i|^2 \psi_i \, \mathrm{d}x\right) \left(\sum_{i=1}^k \int_{\omega_i} \big|\operatorname{Div}\big(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\big)\big|^2 \, \psi_i \, \mathrm{d}x\right), \tag{3.37}$$

that is

$$|I_\varepsilon|^2 \leq \left\|\mathrm{Div}\big(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\big)\right\|^2_{L^2(\Omega)^d} \sum_{i=1}^k \int_{\omega_i} |A_\varepsilon^{-1}\xi_\varepsilon - \nabla v_\varepsilon^i|^2 \, \psi_i \, \mathrm{d}x. \tag{3.38}$$

Using successively (3.8), (3.27) and (3.7), we get that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \sum_{i=1}^k \int_{\omega_i} \left|A_\varepsilon^{-1}\xi_\varepsilon - \nabla v_\varepsilon^i\right|^2 \, \psi_i \, \mathrm{d}x \\
&= \limsup_{\varepsilon\to 0} \sum_{i=1}^k \int_{\omega_i} \left(A_\varepsilon^{-1}\xi_\varepsilon \cdot \xi_\varepsilon - 2\xi_\varepsilon \cdot \nabla v_\varepsilon^i + A_\varepsilon \nabla v_\varepsilon^i \cdot \nabla v_\varepsilon^i\right) \psi_i \, \mathrm{d}x \\
&= \sum_{i=1}^k \int_{\omega_i} \left(A_*^{-1}\xi \cdot \xi - 2\xi \cdot \nabla v^i + A_* \nabla v^i \cdot \nabla v^i\right)\psi_i \, \mathrm{d}x \\
&= \sum_{i=1}^k \int_{\omega_i} \left[A_*\big(A_*^{-1}\xi - \lambda_i\big) \cdot \big(A_*^{-1}\xi - \lambda_i\big)\right]\psi_i \, \mathrm{d}x \leq \beta \sum_{i=1}^k \int_{\omega_i} |A_*^{-1}\xi - \lambda_i|^2 \, \mathrm{d}x \leq \beta\delta.
\end{aligned} \tag{3.39}$$

This combined with (3.31) and (3.38) implies that

$$|I_\varepsilon| = O(\sqrt{\delta}). \tag{3.40}$$

On the other hand, an integration by parts gives

$$\begin{aligned}
J_\varepsilon &= \sum_{i=1}^k \int_{\omega_i} \nabla v_\varepsilon^i \cdot \mathrm{Div}\big(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\big)\psi_i \, \mathrm{d}x \\
&= \sum_{i=1}^k \int_{\omega_i} \nabla v_\varepsilon^i \cdot \mathrm{Div}\big(\psi_i(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}})\big) \, \mathrm{d}x - \sum_{i=1}^k \int_{\omega_i} \nabla v_\varepsilon^i \cdot \big(\Phi_\varepsilon^{\mathrm{T}} - \Phi_\varepsilon\big)\nabla\psi_i \, \mathrm{d}x.
\end{aligned} \tag{3.41}$$

Since $\psi_i(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}})$ is an antisymmetric matrix, we have for any $i = 1, \ldots, k$,

$$\int_{\omega_i} \nabla v_\varepsilon^i \cdot \mathrm{Div}\big(\psi_i(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}})\big) \, \mathrm{d}x = -\sum_{k,l=1}^d \left\langle \frac{\partial^2 v_\varepsilon^i}{\partial x_k \partial x_l}, \, \psi_i(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}})_{k,l} \right\rangle_{\mathscr{C}_c^\infty(\omega_i), \mathscr{D}(\omega_i)} = 0, \tag{3.42}$$

hence

$$J_\varepsilon = \sum_{i=1}^k \int_{\omega_i} \nabla v_\varepsilon^i \cdot \big(\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\big)\nabla\psi_i \, \mathrm{d}x. \tag{3.43}$$

The Cauchy-Schwarz inequality gives

$$|J_\varepsilon|^2 \leq \|\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\|^2_{L^2(\Omega)^{d\times d}} \sup_{1\leq i\leq k} \|\nabla\psi_i\|^2_\infty \sum_{i=1}^k \|\nabla v_\varepsilon^i\|^2_{L^2(\omega_i)}. \tag{3.44}$$

Then, since k and $(\psi_i)_{1\leq i\leq k}$ are independent of ε, there exists $C_\delta > 0$ such that

$$|J_\varepsilon| \leq C_\delta \|\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\|_{L^2(\Omega)^{d\times d}}. \tag{3.45}$$

Combining (3.40) and (3.45) with (3.34), we have, for any $\delta > 0$,

$$\left|\big\langle \mathrm{curl}\big(A_\varepsilon^{-1}\xi_\varepsilon\big), \Phi_\varepsilon\big\rangle_{H^{-1}(\Omega)^{d\times d}, H_0^1(\Omega)^{d\times d}}\right| \leq C_\delta \, \|\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}\|_{L^2(\Omega)^{d\times d}} + O(\sqrt{\delta}). \tag{3.46}$$

Moreover, by the definition of Φ_ε in (3.31) and Rellich's theorem, $\Phi_\varepsilon - \Phi_\varepsilon^{\mathrm{T}}$ converges strongly to 0 in $L^2(\Omega)^{d\times d}$. Therefore, we get that for any $\delta > 0$,

$$\limsup_{\varepsilon\to 0} \left|\big\langle \mathrm{curl}\big(A_\varepsilon^{-1}\xi_\varepsilon\big), \Phi_\varepsilon\big\rangle_{H^{-1}(\Omega)^{d\times d}, H_0^1(\Omega)^{d\times d}}\right| \leq C\sqrt{\delta}, \tag{3.47}$$

which implies (3.9) due to the arbitrariness of $\delta > 0$ and (3.31).

Finally, let us prove that (3.9) implies the equality in (3.8). As $|A_\varepsilon^{-1}\xi_\varepsilon| \le \beta|\xi_\varepsilon|$ is a bounded sequence in $L^2(\Omega)$, the following convergence holds up to a subsequence

$$\eta_\varepsilon := A_\varepsilon^{-1}\xi_\varepsilon \rightharpoonup \eta \quad \text{weakly in } L^2(\Omega)^d. \tag{3.48}$$

By the div-curl lemma and (3.9), we have

$$\eta_\varepsilon \cdot \xi_\varepsilon = A_\varepsilon^{-1}\xi_\varepsilon \cdot \xi_\varepsilon \rightharpoonup \eta \cdot \xi \quad \text{in } \mathscr{D}'(\Omega), \tag{3.49}$$

$$\eta_\varepsilon \cdot A_\varepsilon \nabla v_\varepsilon^\lambda \rightharpoonup \eta \cdot A_*\lambda \quad \text{in } \mathscr{D}'(\Omega). \tag{3.50}$$

Moreover, since A_ε is symmetric, we have

$$\eta_\varepsilon \cdot A_\varepsilon \nabla v_\varepsilon^\lambda = \xi_\varepsilon \cdot \nabla v_\varepsilon^\lambda \rightharpoonup \xi \cdot \lambda \quad \text{in } \mathscr{D}'(\Omega). \tag{3.51}$$

From (3.50) and (3.51), we deduce that for any $\lambda \in \mathbb{R}^d$,

$$\eta \cdot A_*\lambda = A_*\eta \cdot \xi = \xi \cdot \lambda \quad \text{a.e. in } \Omega, \tag{3.52}$$

which implies that

$$\eta = A_*^{-1}\xi \quad \text{a.e. in } \Omega. \tag{3.53}$$

We conclude to the equality in (3.8) combining (3.53) with (3.49). □

3.2 Higher-order terms

In this section, we try to extend the inequality (3.1) when the magneto-resistance is replaced by any term of even-order in the expansion of the perturbed resistivity. We first establish an inequality (opposite to (3.1)) satisfied by the fourth-order term of the resistivity assuming that the Hall matrix is zero. Then, we prove that the positivity is not conserved for even-orders greater than two.

For the sake of simplicity, we lighten the notation of Remark 1.1: for any functional space H, any integer k, any k-linear form $g^k : (\mathbb{R}^3)^k \to H$ and any $h \in \mathbb{R}^3$, the k^{th}-order $g^k(h, \ldots, h)$ is symply denoted $g^k(h)$. Let $n \ge 4$ be an integer. Assume that the conductivity satisfies the regularity condition (2.2) for any multi-index $|\nu| \le n$. As a consequence, the conductivity $\sigma_\varepsilon(h)$, the resistivity $\rho_\varepsilon(h) = \sigma_\varepsilon(h)^{-1}$ and the associated homogenized quantities $\sigma_*(h)$, $\rho_*(h)$ satisfy the n^{th}-order expansions in h

$$\begin{cases} \sigma_\varepsilon(h) = \sigma_\varepsilon + \cdots + \sigma_\varepsilon^n(h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^n), & \rho_\varepsilon(h) = \rho_\varepsilon + \cdots + \rho_\varepsilon^n(h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^n), \\ \sigma_*(h) = \sigma_* + \cdots + \sigma_*^n(h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^n), & \rho_*(h) = \rho_* + \cdots + \rho_*^n(h) + o_{L^\infty(\Omega)^{3\times 3}}(|h|^n), \end{cases} \tag{3.54}$$

where for any $h \in \mathbb{R}^3$, the matrices $\sigma_{\varepsilon/*}^k(h)$, $\rho_{\varepsilon/*}^k(h)$ are symmetric for even k and antisymmetric for odd k. Note that with the notations of Section 2 we have

$$\sigma_{\varepsilon/*}^1(h) = \mathscr{E}(S_{\varepsilon/*}h), \quad \rho_{\varepsilon/*}^1(h) = \mathscr{E}(R_{\varepsilon/*}h), \quad \sigma_{\varepsilon/*}^2(h) = \mathscr{N}_{\varepsilon/*}(h,h), \quad \rho_{\varepsilon/*}^2(h) = \mathscr{M}_{\varepsilon/*}(h,h). \tag{3.55}$$

Similarly to (2.19) and (2.20), by the above regularity condition, the coercivity of $\sigma_\varepsilon(h)$ and the Meyers estimate [16], the potential $U_\varepsilon(h)$ and the corrector $P_\varepsilon(h)$ admit the following n^{th}-order expansions in h,

$$U_\varepsilon(h) = U_\varepsilon^0 + U_\varepsilon^1(h) + \cdots + U_\varepsilon^n(h) + o_{W^{1,p}(\Omega)^3}(|h|^n), \tag{3.56}$$

$$P_\varepsilon(h) = P_\varepsilon^0 + P_\varepsilon^1(h) + \cdots + P_\varepsilon^n(h) + o_{L^p(\Omega)^{3\times 3}}(|h|^n). \tag{3.57}$$

3.2.1 Fourth-order term with zero Hall matrix

We can now state the following result for the fourth-order term:

Proposition 3.1. *Assume that (2.1) and (2.2) for $|\nu| \le 4$ are satisfied and that the norms of σ_ε^2, σ_ε^3 and σ_ε^4 are bounded in $L^\infty(\Omega)$. Then, in the absence of Hall effect (i.e. $\rho_\varepsilon^1 = 0$), we have, for any $h \in O$,*

$$\sigma_*\, \rho_*^4(h)\, \sigma_* \le \lim_{w-L^1(\Omega)} \left(\sigma_\varepsilon P_\varepsilon^0\right)^T \rho_\varepsilon^4(h) \left(\sigma_\varepsilon P_\varepsilon^0\right). \tag{3.58}$$

Moreover, (3.58) is an equality if and only if

$$\operatorname{Curl}\left(\rho_\varepsilon^2(h)\sigma_\varepsilon P_\varepsilon^0\right) \text{ lies in a compact subset of } H^{-1}(\Omega)^{3\times3\times3}. \tag{3.59}$$

Proof. The proof follows the framework of Section 2 and 3. We first establish like in Theorem 2.1 a new expression of the difference of the two terms of (3.58) through relations similar to Proposition 2.1. We then apply Lemma 3.1 to this new expression.

Let $h \in O$. As $\sigma_\varepsilon^1 = 0$, by Proposition 2.1 and (3.55), we have

$$\sigma_*^1 = \rho_\varepsilon^1 = \rho_*^1 = 0. \tag{3.60}$$

Considering the expansion at the fourth-order of $\sigma_\varepsilon(h)\rho_\varepsilon(h) = I_3$, we obtain similarly to the proof of Proposition 2.1, for any $h \in O$,

$$\sigma_\varepsilon\, \rho_\varepsilon^4(h) + \sigma_\varepsilon^1(h)\, \rho_\varepsilon^3(h) + \sigma_\varepsilon^2(h)\, \rho_\varepsilon^2(h) + \sigma_\varepsilon^3(h)\, \rho_\varepsilon^1(h) + \sigma_\varepsilon^4(h)\, \rho_\varepsilon = 0, \tag{3.61}$$

which gives, by (3.60),

$$\sigma_\varepsilon\, \rho_\varepsilon^4(h)\, \sigma_\varepsilon = -\, \sigma_\varepsilon^2(h)\, \rho_\varepsilon^2(h)\, \sigma_\varepsilon - \sigma_\varepsilon^4(h). \tag{3.62}$$

Using again (3.60) with (3.55), (2.10) can be rewritten, for any $h \in O$,

$$\sigma_\varepsilon^2(h) = \sigma_\varepsilon\, \rho_\varepsilon^2(h)\, \sigma_\varepsilon. \tag{3.63}$$

Combining (3.62) with (3.63), we obtain

$$\sigma_\varepsilon\, \rho_\varepsilon^4(h)\, \sigma_\varepsilon = \sigma_\varepsilon^2(h)\, \sigma_\varepsilon^{-1}\, \sigma_\varepsilon^2(h) - \sigma_\varepsilon^4(h). \tag{3.64}$$

Similarly, we have for the homogenized quantities

$$\sigma_*\, \rho_*^4(h)\, \sigma_* = \sigma_*^2(h)\, \sigma_*^{-1}\, \sigma_*^2(h) - \sigma_*^4(h). \tag{3.65}$$

Taking into account the expansions (3.54) and (3.57), we have (writing only the second and the fourth-order terms) like in (2.23)

$$\begin{aligned} \sigma_\varepsilon(h)P_\varepsilon(h) = \cdots &+ \left(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^1 P_\varepsilon^1(h) + \sigma_\varepsilon^2(h)P_\varepsilon^0\right) + \cdots \\ &+ \left(\sigma_\varepsilon P_\varepsilon^4(h) + \sigma_\varepsilon^1(h)P_\varepsilon^3(h) + \sigma_\varepsilon^2(h)P_\varepsilon^2(h) + \sigma_\varepsilon^3(h)P_\varepsilon^1(h) + \sigma_\varepsilon^4(h)P_\varepsilon^0\right) + o_{L^p(\Omega)^{3\times3}}(|h|^4), \end{aligned} \tag{3.66}$$

that is by (3.60)

$$\begin{aligned} \sigma_\varepsilon(h)P_\varepsilon(h) \ &= \ \cdots + \left(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h)P_\varepsilon^0\right) + \cdots \\ &\quad + \left(\sigma_\varepsilon P_\varepsilon^4(h) + \sigma_\varepsilon^2(h)P_\varepsilon^2(h) + \sigma_\varepsilon^3(h)P_\varepsilon^1(h) + \sigma_\varepsilon^4(h)P_\varepsilon^0\right) + o_{L^p(\Omega)^{3\times3}}(|h|^4). \end{aligned} \tag{3.67}$$

By virtue of Remark 1.1, using the properties (2.16)-(2.18) satisfied by the corrector $P_\varepsilon(h)$ in the expansions (3.57), (3.67) and (3.54), we get that

$$\left\{ \begin{array}{rcll} P_\varepsilon^4(h) & \rightharpoonup & 0 & \text{weakly in } L^p(\Omega)^{3\times3}, \\ \sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h)P_\varepsilon^0 & \rightharpoonup & \sigma_*^2(h) & \text{weakly in } L^p(\Omega)^{3\times3}, \\ \sigma_\varepsilon P_\varepsilon^4(h) + \sigma_\varepsilon^2(h)P_\varepsilon^2(h) + \sigma_\varepsilon^3(h)P_\varepsilon^1(h) + \sigma_\varepsilon^4(h)P_\varepsilon^0 & \rightharpoonup & \sigma_*^4(h) & \text{weakly in } L^p(\Omega)^{3\times3}. \end{array} \right. \tag{3.68}$$

and

$$\begin{cases} \mathrm{Div}\big(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h) P_\varepsilon^0\big) &= \mathrm{Div}\big(\sigma_*^2(h)\big), \\ \mathrm{Div}\big(\sigma_\varepsilon P_\varepsilon^4(h) + \sigma_\varepsilon^2(h) P_\varepsilon^2(h) + \sigma_\varepsilon^3(h) P_\varepsilon^1(h) + \sigma_\varepsilon^4(h) P_\varepsilon^0\big) &= \mathrm{Div}\big(\sigma_*^4(h)\big) \end{cases} \quad \text{in } \mathscr{D}'(\Omega)^{3\times 3}. \tag{3.69}$$

Moreover, from $\sigma_\varepsilon^1 = 0$, (3.60) and (2.25) $\sigma_\varepsilon P_\varepsilon^1(h)$ is a divergence free function. Then the div-curl lemma implies that $\big(\sigma_\varepsilon P_\varepsilon^1(h)\big)^{\mathrm{T}} P_\varepsilon^1(h)$ converges to 0 in $L^{p/2}(\Omega)^{3\times 3}$. This combined with $\sigma_\varepsilon \in \mathcal{M}(\alpha, \beta; \Omega)$, we get that

$$P_\varepsilon^1(h) \longrightarrow 0 \quad \text{strongly in } L^p(\Omega)^{3\times 3}. \tag{3.70}$$

Taking into account (2.24) and (3.69), the div-curl lemma implies the convergence

$$\big(P_\varepsilon^0\big)^{\mathrm{T}}\big(\sigma_\varepsilon P_\varepsilon^4(h) + \sigma_\varepsilon^2(h) P_\varepsilon^2(h) + \sigma_\varepsilon^3(h) P_\varepsilon^1(h) + \sigma_\varepsilon^4(h) P_\varepsilon^0\big) \rightharpoonup \sigma_*^4(h) \quad \text{weakly in } L^{p/2}(\Omega)^{3\times 3}. \tag{3.71}$$

Moreover by (2.24), (3.68), (2.25) and the symmetry of σ_ε the div-curl lemma yields

$$\big(P_\varepsilon^0\big)^{\mathrm{T}}\big(\sigma_\varepsilon P_\varepsilon^4(h)\big) = \big(\sigma_\varepsilon P_\varepsilon^0\big)^{\mathrm{T}} P_\varepsilon^4(h) \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega)^{3\times 3}. \tag{3.72}$$

Hence, combining (3.70) and (3.72) in (3.71) we obtain

$$\sigma_*^4(h) = \lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^2(h) P_\varepsilon^2(h) + \big(P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^4(h) P_\varepsilon^0\Big]. \tag{3.73}$$

Taking into account (2.17), and (3.69) the div-curl lemma implies that

$$\big(P_\varepsilon^2(h)\big)^{\mathrm{T}}\big(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h) P_\varepsilon^0\big) \rightharpoonup 0 \quad \text{weakly in } L^{p/2}(\Omega)^{3\times 3}, \tag{3.74}$$

hence

$$\begin{aligned} &\lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h) P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^{-1}\big(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h) P_\varepsilon^0\big) \\ &= \lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^2(h) P_\varepsilon^2(h) + \big(P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^2(h) \sigma_\varepsilon^{-1} \sigma_\varepsilon^2(h) P_\varepsilon^0\Big]. \end{aligned} \tag{3.75}$$

Combining the equalities (3.64), (3.65) with (3.73), (3.75) and the symmetry of σ_ε and $\sigma_\varepsilon^2(h)$, we obtain that

$$\begin{aligned} &\sigma_*\, \rho_*^4(h)\, \sigma_* - \lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon P_\varepsilon^0\big)^{\mathrm{T}} \rho_\varepsilon^4(h) \big(\sigma_\varepsilon P_\varepsilon^0\big) \\ &= \sigma_*^2(h)\, (\sigma_*)^{-1}\, \sigma_*^2(h) - \sigma_*^4(h) - \lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^2(h) \sigma_\varepsilon^{-1} \sigma_\varepsilon^2(h) P_\varepsilon^0 - \big(P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^4(h) P_\varepsilon^0\Big] \\ &= \sigma_*^2(h)\, (\sigma_*)^{-1}\, \sigma_*^2(h) - \lim_{w-L^1(\Omega)} \Big[\big(P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^2(h) \sigma_\varepsilon^{-1} \sigma_\varepsilon^2(h) P_\varepsilon^0 + \big(P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^2(h) P_\varepsilon^2(h)\Big] \\ &= \sigma_*^2(h)\, (\sigma_*)^{-1}\, \sigma_*^2(h) - \lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h) P_\varepsilon^0\big)^{\mathrm{T}} \sigma_\varepsilon^{-1}\big(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h) P_\varepsilon^0\big). \end{aligned} \tag{3.76}$$

Let $\lambda \in \mathbb{R}^3$. We apply Lemma 3.1 with $A_\varepsilon := \sigma_\varepsilon$, $\xi_\varepsilon := \big(\sigma_\varepsilon P_\varepsilon^2(h) + \sigma_\varepsilon^2(h) P_\varepsilon^0\big)\lambda$ which has a compact divergence by (3.69) and converges weakly in $L^p(\Omega)^{3\times 3}$ to $\xi := \sigma_*^2(h)\lambda$ by (3.68). Thus, with the notation of Lemma 3.1, we have

$$\zeta = \lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon^2(h) P_\varepsilon^0 + \sigma_\varepsilon P_\varepsilon^2(h)\big)^{\mathrm{T}} \sigma_\varepsilon^{-1}\big(\sigma_\varepsilon^2(h) P_\varepsilon^0 + \sigma_\varepsilon P_\varepsilon^2(h)\big)\lambda \cdot \lambda. \tag{3.77}$$

As $\sigma_*^2(h)$ is symmetric, it follows that for any $\lambda \in \mathbb{R}^3$,

$$\lim_{w-L^1(\Omega)} \big(\sigma_\varepsilon^2(h) P_\varepsilon^0 + \sigma_\varepsilon P_\varepsilon^2(h)\big)^{\mathrm{T}} \sigma_\varepsilon^{-1}\big(\sigma_\varepsilon^2(h) P_\varepsilon^0 + \sigma_\varepsilon P_\varepsilon^2(h)\big)\lambda \cdot \lambda \geq \sigma_*^2(h) \sigma_*^{-1} \sigma_*^2(h) \lambda \cdot \lambda. \tag{3.78}$$

Using the fact that $P_\varepsilon^2(h)$ is a gradient and the equality (3.63), (3.78) is an equality if and only if

$$\mathrm{curl}\,\big(A_\varepsilon^{-1}\xi_\varepsilon\big) = \mathrm{curl}\,\big(\sigma_\varepsilon^{-1}\sigma_\varepsilon^2(h) P_\varepsilon^0 \lambda\big) = -\,\mathrm{curl}\,\big(\rho_\varepsilon^2(h)\sigma_\varepsilon P_\varepsilon^0 \lambda\big) \tag{3.79}$$

lies in a compact subset of $H^{-1}(\Omega)^{3\times 3}$. This concludes the proof due to the arbitrariness of λ. $\square$

3.2.2 An example with changes of sign

In this section we build a rank-one laminate which shows that the inequality (3.1) (or its inverse) cannot be extended to higher even-order terms.

Let $p \in \mathbb{N}^*$. Define the perturbed conductivity

$$\sigma(h) := \chi\,\sigma_1(h) + (1-\chi)\,\sigma_2(h), \tag{3.80}$$

where χ is a characteristic function. For $i = 1, 2$, the conductivities $\sigma_i(h)$ belong to $\mathscr{M}(\alpha, \beta; \Omega)$ and the resistivities $\rho_i(h) = \sigma_i(h)^{-1}$ satisfy the n^{th}-order expansions in h

$$\sigma_i(h) = \sigma_i + \cdots + \sigma_i^{2p}(h) + o(|h|^{2p}), \quad \rho_i(h) = \rho_i + \cdots + \rho_i^{2p}(h) + o(|h|^{2p}), \tag{3.81}$$

where for any $h \in \mathbb{R}^3$, the matrices $\sigma_i^k(h)$, $\rho_i^k(h)$ are symmetric for even k and antisymmetric for odd k.

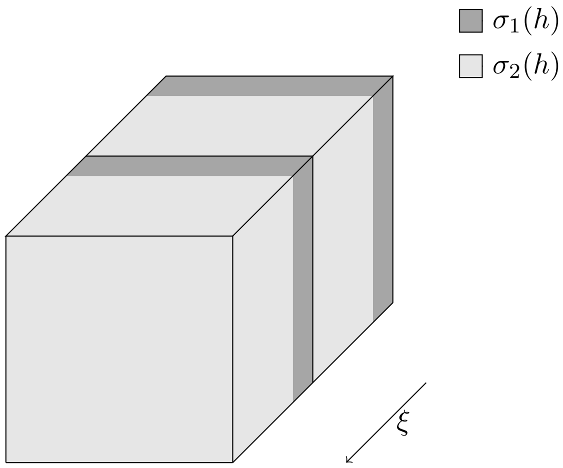

Figure 3.1: A three-dimensional rank-one laminate

We have the following result:

Proposition 3.2. *For any even integer $p \geq 2$, there exists a rank-one laminate such that for any h the matrix*

$$\mathscr{D}^{(2p)}(h) := \sigma_*\rho_*^{(2p)}(h)\sigma_* - \Big\langle \big(\sigma P^0\big)^T \rho^{(2p)}(h)\big(\sigma P^0\big)\Big\rangle \tag{3.82}$$

is neither non-positive, nor non-negative.

Proof. We consider the particular case of (3.80) where the magnetic field is $h = h_3\, e_3$, χ is a 1-periodic function only depending on x_1, and

$$\sigma_1(h) = \theta^{-1} I_3 + \theta^{-1}\mathscr{E}(h), \quad \sigma_2(h) = \alpha_2\, I_3 + \mathscr{E}(h), \quad \text{with} \quad \theta := \langle\chi\rangle \in (0,1), \quad \alpha_2 > 0. \tag{3.83}$$

The laminate corrector $P(h)$ is explicitly given by (see, *e.g.*, [6])

$$P(h) = \chi\, P_1(h) + (1-\chi)\, P_2(h), \tag{3.84}$$

where for $i = 1, 2$,

$$P_i(h) = I_3 + \frac{(1-\theta)^{(2-i)}\,(-\theta)^{i-1}}{1-\theta+\theta^2\alpha_2}\begin{pmatrix} \theta\,\alpha_2 - 1 & (1-\theta)\,h_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.85}$$

The homogenized conductivity is defined by

$$\sigma_*(h) = \big\langle \chi\,\sigma_1(h)P_1(h) + (1-\chi)\,\sigma_2(h)P_2(h)\big\rangle, \tag{3.86}$$

which yields

$$\sigma_*(h) = \begin{pmatrix} \dfrac{\alpha_2}{1-\theta+\theta^2\,\alpha_2} & -\dfrac{1-\theta+\theta\,\alpha_2}{1-\theta+\theta^2\,\alpha_2}\,h_3 & 0 \\[2ex] \dfrac{1-\theta+\theta\,\alpha_2}{1-\theta+\theta^2\,\alpha_2}\,h_3 & 1+(1-\theta)\,\alpha_2+\dfrac{(1-\theta)^3}{1-\theta+\theta^2\,\alpha_2}\,h_3^2 & 0 \\[2ex] 0 & 0 & 1+(1-\theta)\,\alpha_2 \end{pmatrix}. \tag{3.87}$$

Inverting this matrix, we obtain the homogenized resistivity

$$\rho_*(h) = \begin{pmatrix} \dfrac{b\,(1-\theta+\theta^2\,\alpha_2)^2+(1-\theta)^3\,h_3^2}{b\,\alpha_2+c\,h_3^2} & \dfrac{1-\theta+\theta\,\alpha_2}{b\,\alpha_2+c\,h_3^2} & 0 \\[2ex] -\dfrac{1-\theta+\theta\,\alpha_2}{b\,\alpha_2+c\,h_3^2} & \dfrac{\alpha_2}{b\,\alpha_2+c\,h_3^2} & 0 \\[2ex] 0 & 0 & b^{-1} \end{pmatrix}, \text{ where } \begin{cases} b := 1+(1-\theta), \\ c := 1-\theta+\alpha_2. \end{cases} \tag{3.88}$$

Expanding the quantities $\rho_i(h) = \sigma_i(h)^{-1}$, we obtain the expressions

$$\rho_1^{(2p)}(h) = (-1)^p\,\theta\,h_3^{2p}\,K \quad\text{and}\quad \rho_2^{(2p)}(h) = (-1)^p\,\frac{h_3^{(2p)}}{\alpha_2^{2p+1}}\,K, \quad\text{where } K = \left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&0 \end{smallmatrix}\right). \tag{3.89}$$

Using (3.83)-(3.89), we can compute $\mathscr{D}^{(2p)}(h)$. All the coefficients in the matrix $\mathscr{D}^{(2p)}(h)$ are zero except the entries

$$\begin{cases} \mathscr{D}^{(2p)}(h)_{1,1} = \dfrac{(-1)^p}{(1-\theta+\theta^2\,\alpha_2)^2}\left[\dfrac{(1-\theta+\theta\alpha_2)^2\,(1-\theta+\alpha_2)^{p-1}}{\alpha_2^{p-1}\,\big(1+(1-\theta)\alpha_2\big)^p} - \dfrac{1-\theta}{\alpha_2^{2p-1}} - (\theta\,\alpha_2)^2\right] h_3^{2p}, \\[3ex] \mathscr{D}^{(2p)}(h)_{2,2} := (-1)^p\left[\dfrac{(1-\theta+\alpha_2)^p}{\alpha_2^p\,\big(1+(1-\theta)\alpha_2\big)^{p-1}} - 1 - \dfrac{1-\theta}{\alpha_2^{2p-1}}\right] h_3^{2p}. \end{cases} \tag{3.90}$$

As $p \geq 2$, passing to the limit in (3.90) successively when $\theta \to 0$ and $\alpha_2 \to \infty$, we obtain that

$$\lim_{\theta\to 0} \mathscr{D}^{(2p)}(h) \underset{\alpha_2\to\infty}{\sim} (-1)^p\,h_3^{2p} \begin{pmatrix} \dfrac{1}{\alpha_2^p} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.91}$$

Finally, when θ is small enough and α_2 large enough, the matrix $\mathscr{D}^{(2p)}(h)$ has a positive eigenvalue, and a negative eigenvalue, which concludes the proof. $\square$

Remark 3.3. The case $p = 1$ confirms Theorem 3.1. Indeed, we have

$$\mathscr{D}^{(2)}(h) = \frac{(1+(1-\theta)\alpha_2)^{-1}}{(1-\theta+\theta^2\,\alpha_2)^2}\left[\left(\frac{1-\theta}{\alpha_2}+\theta\,\big(\theta\alpha_2^2\big)\right)\big((1-\theta)\alpha_2+\theta\,\theta^{-1}\big) - (1-\theta+\theta\alpha_2)^2\right] h_3^2\,(e_1\otimes e_1),$$

which is positive by the Cauchy-Schwarz inequality. This formula is a particular case of formula (4.6) below.

4 Case of equality for a few periodic structures

In this section we consider various periodic microstructures in the case of equality for (3.3). On the one hand, Section 4.1 provides an explicit expression of the difference between the two terms of (3.3) for layered structures and thus the different cases of equality. On the other hand, Section 4.2 only provides the cases of equality for columnar structures: condition (3.4) would have consequences on the Hall matrix and the conductivity of the microstructure. We use the notations of Section 2.2.2.

More precisely, we study the consequences of $D(h,h)=0$ in (1.10). For a given averaged-value $\lambda\in\mathbb{R}^3$ of the electric field $e=P^0\lambda$ in a composite conductor, we have the relations for the local current and the averaged-value of the current (see Remark 3.1)

$$j=\sigma e=\sigma P^0\lambda,\quad \langle j\rangle=\sigma_*\langle e\rangle=\sigma_*\lambda. \tag{4.1}$$

We set

$$\mathscr{D}(h,h):=\sigma_*\,\mathscr{M}_*(h,h)\,\sigma_*-\left\langle\left(\sigma P^0\right)^{\mathrm{T}}\mathscr{M}(h,h)\left(\sigma P^0\right)\right\rangle, \tag{4.2}$$

so that, by the symmetry of σ_*, it follows that

$$D(h,h)=\mathscr{D}(h,h)\lambda\cdot\lambda. \tag{4.3}$$

4.1 Periodic layered structures

In this section, we establish for a periodic layered structure depending on a direction $\xi\in\mathbb{R}^3$, $|\xi|=1$, an exact formula for the difference between the effective magneto-resistance and the averaged local magneto-resistance.

Let $\sigma(h)$ be a perturbed conductivity in $\mathcal{M}(\alpha,\beta;\Omega)$ only depending on $\xi\cdot y$, and satisfying the expansion

$$\sigma(h)(y)=a(\xi\cdot y)\,I_3+s(\xi\cdot y)\,\mathscr{E}(h)+\mathscr{N}(h,h)(\xi\cdot y)+o(|h|^2),\quad \text{for a.e. } y\in Y, \tag{4.4}$$

where $a:\mathbb{R}\to[\alpha,\beta]$ and $s:\mathbb{R}\to\mathbb{R}$ are 1-periodic functions. By Proposition 2.1, we have

$$R=r\,I_3=-\frac{s}{a^2}\,I_3. \tag{4.5}$$

Considering the expansions (2.39)-(2.41), we can state a result precising Corollary 3.1:

Proposition 4.1. *Consider a conductivity $\sigma(h)$ satisfying (4.4) and the matrix-valued function $\mathscr{D}(h,h)$ defined by (4.2).*

- *When h is not parallel to ξ, we have*

$$O(h)^T\mathscr{D}(h,h)\,O(h)=\begin{pmatrix} d_1\,|h\times\xi|^2 & d_3\,(h\cdot\xi)|h\times\xi| & 0\\ d_3\,(h\cdot\xi)|h\times\xi| & d_2\,(h\cdot\xi)^2 & 0\\ 0 & 0 & d_2\,(h\cdot\xi)^2\end{pmatrix}, \tag{4.6}$$

where $O(h)$ is the change-of-basis matrix from the canonical basis to

$$\hat{\mathscr{B}}=\left(\xi,\frac{\xi\times(\xi\times h)}{|h\times\xi|},\frac{h\times\xi}{|h\times\xi|}\right),\quad \text{and}\quad \begin{cases} d_1:=\left\langle a^{-1}\right\rangle^{-2}\left[\left\langle a\,r^2\right\rangle-\langle a\rangle^{-1}\langle a\,r\rangle^2\right],\\ d_2:=\left\langle a^3\,r^2\right\rangle-\langle a\rangle^{-1}\langle a^2\,r\rangle^2,\\ d_3:=\left\langle a^{-1}\right\rangle^{-1}\left[\left\langle a^2\,r^2\right\rangle-\langle a\rangle^{-1}\langle a\,r\rangle\,\langle a^2\,r\rangle\right].\end{cases} \tag{4.7}$$

- *When h is parallel to ξ, we have*

$$O^{\mathrm{T}}\mathscr{D}(h,h)\,O = d_2\,(h\cdot\xi)^2\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}, \tag{4.8}$$

where O is the change-of-basis matrix from the canonical basis to an orthonormal basis $\hat{\mathscr{B}} = (\xi, u, v)$, for suitable $u, v \in \mathbb{R}^3$.

Moreover, $\mathscr{D}(h,h) = 0$ if and only if one of the following conditions holds:

- *$h = 0$;*
- *$h \neq 0$ is orthogonal to ξ, and r is a constant;*
- *$h \neq 0$ is parallel to ξ, and ar is a constant;*
- *$h \neq 0$ is neither parallel to ξ, nor orthogonal to ξ, and ar, r are constant.*

Proof.
First case: $h \neq 0$ is not parallel to ξ. By Corollary 2.1 we have

$$\mathscr{D}(h,h) = \left\langle \big(\mathscr{E}(Sh)P^0 + \sigma P^1(h)\big)^{\mathrm{T}}\sigma^{-1}\big(\mathscr{E}(Sh)P^0 + \sigma P^1(h)\big)\right\rangle - \mathscr{E}(S_*h)^{\mathrm{T}}\sigma_*^{-1}\mathscr{E}(S_*h). \tag{4.9}$$

For the sake of simplicity denote $O := O(h)$. Denoting by $\hat{\cdot}$ the quantities with respect to the new basis $\hat{\mathscr{B}}$, we have the following change-of-basis formulas respectively for the system of coordinates, the local conductivity, the zero and first-order terms in the expansion of the corrector and the local S-matrix defined by (2.35):

$$\begin{cases} \hat{y} = O^{\mathrm{T}}y, & \hat{h} = O^{\mathrm{T}}h, \\ \hat{\sigma}(\hat{h}) = O^{\mathrm{T}}\sigma(h)\,O, & \hat{\sigma} = O^{\mathrm{T}}\sigma\,O, \\ \hat{S} = O^{\mathrm{T}}S\,O & \text{(as a consequence of (2.14))}, \\ \hat{P}^0 = O^{\mathrm{T}}P^0\,O, & \hat{P}^1(\hat{h}) = O^{\mathrm{T}}P^1(h)\,O, \end{cases} \tag{4.10}$$

where the last equality is a consequence of the relation

$$\forall\,\lambda\in\mathbb{R}^3,\quad \hat{P}(\hat{h})\hat{\lambda} = O^{\mathrm{T}}P(h)\lambda = O^{\mathrm{T}}P(h)\,O\,O^{\mathrm{T}}\lambda = O^{\mathrm{T}}P(h)\,O\,\hat{\lambda},\quad \text{with } \hat{\lambda} = O^{\mathrm{T}}\lambda. \tag{4.11}$$

Due to Lemma 38 of [21] for the homogenized conductivity and to (2.14) for the homogenized S-matrix defined by (2.39), we have

$$\hat{\sigma}_*(\hat{h}) = O^{\mathrm{T}}\sigma_*(h)\,O,\quad \hat{\sigma}_* = O^{\mathrm{T}}\sigma_*\,O \quad\text{and}\quad \hat{S}_* = O^{\mathrm{T}}S_*\,O. \tag{4.12}$$

From these relations and (4.9) it is easy to check that the difference term $\mathscr{D}(h,h)$ defined by (4.2) satisfies the relation

$$\hat{\mathscr{D}}(\hat{h},\hat{h}) = O^{\mathrm{T}}\mathscr{D}(h,h)\,O, \tag{4.13}$$

where

$$\hat{\mathscr{D}}(\hat{h},\hat{h}) := \left\langle \big(\mathscr{E}(\hat{S}\,\hat{h})\hat{P}^0 + \hat{\sigma}\hat{P}^1(\hat{h})\big)^{\mathrm{T}}\hat{\sigma}^{-1}\big(\mathscr{E}(\hat{S}\,\hat{h})\hat{P}^0 + \hat{\sigma}\hat{P}^1(\hat{h})\big)\right\rangle - \mathscr{E}(\hat{S}_*\,\hat{h})^{\mathrm{T}}\hat{\sigma}_*^{-1}\mathscr{E}(\hat{S}_*\,\hat{h}). \tag{4.14}$$

Let us now compute $\hat{\mathscr{D}}(\hat{h},\hat{h})$ in (4.14). We have

$$h = (h\cdot\xi, -|h\times\xi|, 0)^{\mathrm{T}} = (\hat{h}_1, \hat{h}_2, 0)^{\mathrm{T}}. \tag{4.15}$$

By isotropy, we have $\hat\sigma = a\, I_3$ and $\hat S = s\, I_3$. By the uniqueness of the solution of problem (2.37)-(2.38), we have

$$\hat P^0 = \begin{pmatrix} \dfrac{\langle a^{-1}\rangle^{-1}}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat P^1(\hat h) = \begin{pmatrix} 0 & 0 & -\dfrac{s - \langle a^{-1}\rangle^{-1}\left\langle \dfrac{s}{a}\right\rangle}{a}\,\hat h_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

Hence, combining the equality (4.16) with the classical periodic homogenization formula (see, *e.g.*, [18] for more details)

$$\hat\sigma_* = \langle \hat\sigma\, \hat P^0\rangle = \begin{pmatrix} \langle a^{-1}\rangle^{-1} & 0 & 0 \\ 0 & \langle a\rangle & 0 \\ 0 & 0 & \langle a\rangle \end{pmatrix}. \tag{4.17}$$

Moreover, we have

$$\mathscr{E}(\hat S\,\hat h)\hat P^0 + \hat\sigma\,\hat P^1(\hat h) = s\,\mathscr{E}(\hat h)\,\hat P^0 + a\,\hat P^1(\hat h) = \begin{pmatrix} 0 & 0 & \langle \sigma^{-1}\rangle^{-1}\left\langle \dfrac{s}{a}\right\rangle \hat h_2 \\ 0 & 0 & -s\,\hat h_1 \\ -\langle \sigma^{-1}\rangle^{-1}\dfrac{s}{a}\,\hat h_2 & s\,\hat h_1 & 0 \end{pmatrix}. \tag{4.18}$$

Also by Corollary 2.1, we obtain that

$$\hat S_* = \langle \mathrm{Cof}(\hat P^0)\hat S\rangle = \begin{pmatrix} \langle s\rangle & 0 & 0 \\ 0 & \langle a^{-1}\rangle^{-1}\left\langle \dfrac{s}{a}\right\rangle & 0 \\ 0 & 0 & \langle a^{-1}\rangle^{-1}\left\langle \dfrac{s}{a}\right\rangle \end{pmatrix}, \tag{4.19}$$

and

$$\mathscr{E}(\hat S_*\,\hat h) = \begin{pmatrix} 0 & 0 & \langle a^{-1}\rangle^{-1}\left\langle \dfrac{s}{a}\right\rangle \hat h_2 \\ 0 & 0 & -\langle s\rangle\,\hat h_1 \\ -\langle a^{-1}\rangle^{-1}\left\langle \dfrac{s}{a}\right\rangle \hat h_2 & \langle s\rangle\,\hat h_1 & 0 \end{pmatrix}. \tag{4.20}$$

Putting (4.18)-(4.20) in (4.14), we get that

$$\hat{\mathscr{D}}(\hat h,\hat h) = \begin{pmatrix} d_1\,\hat h_2^2 & -d_3\,\hat h_1\,\hat h_2 & 0 \\ -d_3\,\hat h_1\,\hat h_2 & d_2\,\hat h_1^2 & 0 \\ 0 & 0 & d_2\,\hat h_1^2 \end{pmatrix}, \tag{4.21}$$

where

$$\begin{cases} d_1 := \langle a^{-1}\rangle^{-2}\left[\left\langle \dfrac{s^2}{a^3}\right\rangle - \langle a\rangle^{-1}\left\langle \dfrac{s}{a}\right\rangle^2\right], \\[2mm] d_2 := \left\langle \dfrac{s^2}{a}\right\rangle - \langle a\rangle^{-1}\langle s\rangle^2, \\[2mm] d_3 := \langle a^{-1}\rangle^{-1}\left[\left\langle \dfrac{s^2}{a^2}\right\rangle - \langle a\rangle^{-1}\left\langle \dfrac{s}{a}\right\rangle\langle s\rangle\right]. \end{cases} \tag{4.22}$$

We deduce (4.6) from (4.5) and (4.15).

Second case: $h \neq 0$ is parallel to ξ. Then, we have $P_1(h) = 0$, and the computations are quite similar.

Cases of equality. When $h \neq 0$ is not parallel to ξ but orthogonal to ξ, by (4.6) $\mathscr{D}(h,h)=0$ implies that $d_1=0$. Thus, the equality

$$\langle a\, r^2\rangle = \langle a\rangle^{-1} \langle a\, r\rangle^2 \tag{4.23}$$

can be regarded as the case of equality in the Cauchy-Schwarz inequality satisfied by the functions $\sqrt{a}$ and $\sqrt{a}\, r$ in $L^2\big([0,1]\big)$. Therefore, $\sqrt{a}$ is proportional to $\sqrt{a}\, r$, hence r is constant. The converse is immediate. The other cases are similar. □

4.2 Periodic columnar structures

4.2.1 The general case

In this section, we consider columnar isotropic structures in the direction y_3. More precisely, the Y-periodic conductivity $\sigma(h)$ of (2.35) only depends on $y'=(y_1,y_2)$ with

$$\begin{cases} \sigma(0) := \sigma(y')\, I_3, & \sigma \in L^\infty_\sharp\big((0,1)^2;[\alpha,\beta]\big),\\ S := s(y')\, I_3, & s \in L^\infty_\sharp\big((0,1)^2;\mathbb{R}\big),\\ \mathscr{N}(h,h) := \mathscr{N}(h,h)(y'), & \mathscr{N}(h,h) \in L^\infty_\sharp\big((0,1)^2;\mathbb{R}^{3\times 3}_s\big). \end{cases} \tag{4.24}$$

Consequently, by Proposition 2.1 the expansion (2.36) of $\rho(h)$ satisfies

$$\begin{cases} \rho = \sigma(y')^{-1}\, I_3, & \\ R = r(y')\, I_3, & \text{with } r=-\sigma^{-2}\, s,\\ \mathscr{M}(h,h) := \mathscr{M}(h,h)(y'), & \mathscr{M}(h,h) \in L^\infty_\sharp\big((0,1)^2;\mathbb{R}^{3\times 3}_s\big). \end{cases} \tag{4.25}$$

We have the following result:

Proposition 4.2. *Consider a conductivity $\sigma(h)$ satisfying (4.24) and set $h'=(h_1,h_2)$. Assume that*

$$r^{-1} \in L^1(Y) \quad \textit{and} \quad \big\langle(\sigma\, r)^{-1}\big\rangle \neq 0. \tag{4.26}$$

Then, $\mathscr{D}(h,h)=0$ (see (4.2)) is an equality if and only if one of the following conditions holds

- $h=0$;
- *$h'=0$, $h_3\neq 0$, and the Hall coefficient r is constant;*
- *$h'\neq 0$, $h_3=0$ and there exist two positive functions f, g in $L^\infty(\mathbb{R})$, with f^{-1}, g^{-1} in $L^\infty(\mathbb{R})$, which are h_i-periodic for $i=1,2$, and a constant C such that*

  $$\sigma(y') = f(h'\cdot y')\; g(Jh'\cdot y') \quad \textit{and} \quad r(y') = \frac{C}{f(h'\cdot y')} \quad \textit{a.e. } y'\in(0,1)^2, \tag{4.27}$$

  where $J := \left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$;
- *$h'\neq 0$, $h_3\neq 0$, the Hall coefficient r is constant, and there exists a function g in $L^\infty(\mathbb{R})$ with g^{-1} in $L^\infty(\mathbb{R})$ which is h_i-periodic for $i=1,2$, such that*

  $$\sigma(y') = \; g(Jh'\cdot y') \quad \textit{a.e. } y'\in(0,1)^2. \tag{4.28}$$

Moreover, when $h_1h_2\neq 0$ and $h_1/h_2\notin\mathbb{Q}$, σ and r are constant.

Remark 4.1. The case $(h_1,h_2)=0$ corresponds to the two-dimensional case in [5] (Theorem 2.4). In the case $(h_1,h_2)\neq(0,0)$, $h_3=0$, f and g are not unique. For example f and g can be chosen such that $\langle f^{-1}\rangle = 1$ to ensure the uniqueness.

Proof of Proposition 4.2. We work in the orthonormal basis (f_1, f_2, e_3) defined by

$$(f_1, f_2) := \begin{cases} \left(\dfrac{h'}{|h'|}, \dfrac{Jh'}{|h'|}\right) & \text{if } h' \neq (0,0), \\ (e_1, e_2) & \text{if } h' = (0,0). \end{cases} \tag{4.29}$$

In the new basis, we have $h = |h'| f_1 + h_3 e_3$. The associated system of coordinates is given by

$$\begin{cases} z_1 := \dfrac{h_1 y_1 + h_2 y_2}{|h'|}, \\ z_2 := \dfrac{h_1 y_2 - h_2 y_1}{|h'|}, & \text{if } h' \neq 0, \quad \text{and} \quad z = y, \text{ if } h' = 0. \\ z_3 := y_3, \end{cases} \tag{4.30}$$

We denote for $i = 1, 2$, $P^0 f_i := \nabla u^i$, $P^0 e_3 := \nabla u^3$ and for $i = 1, 2, 3$, $v^i(z) = u^i(y)$.

Since the gradient, the divergence and the curl are invariant by a change of orthonormal right-handed basis, by (3.4) we have for any $i = 1, 2, 3$,

$$0 = \operatorname{curl}\left((\sigma r)\mathscr{E}(h)\nabla u^i\right) = \begin{pmatrix} |h'|\,\partial_{z_2}\left((\sigma r)\,\partial_{z_2} v^i\right) - \partial_{z_3}\left((\sigma r)\left(h_3\,\partial_{z_1} v^i - |h'|\partial_{z_3} v^i\right)\right) \\ -\,h_3\,\partial_{z_3}\left((\sigma r)\,\partial_{z_2} v^i\right) - |h'|\,\partial_{z_1}\left((\sigma r)\,\partial_{z_2} v^i\right) \\ \partial_{z_1}\left((\sigma r)\left(h_3\partial_{z_1} v^i - |h'|\partial_{z_3} v^i\right)\right) + h_3\,\partial_{z_2}\left((\sigma r)\,\partial_{z_2} v^i\right) \end{pmatrix}. \tag{4.31}$$

As v^i, σ, r are independent of z_3, (4.31) reads as

$$\begin{pmatrix} |h'|\,\partial_{z_2}\left((\sigma r)\,\partial_{z_2} v^i\right) \\ |h'|\,\partial_{z_1}\left((\sigma r)\,\partial_{z_2} v^i\right) \\ h_3\,\partial_{z_1}\left((\sigma r)\,\partial_{z_1} v^i\right) + h_3\,\partial_{z_2}\left((\sigma r)\,\partial_{z_2} v^i\right) \end{pmatrix} = 0, \quad \text{for } i = 1, 2. \tag{4.32}$$

First case: $h' = 0$ and $h_3 \neq 0$. We are led to the two-dimensional case of [4] with $h = h_3\, e_3$. The key ingredient is the positivity of the determinant of the corrector P_0 due to Alessandrini and Nesi [1].

Second case: $h' \neq 0$ and $h_3 = 0$. Without loss of generality, we can assume that $|h'| = 1$. The two first equalities of (4.32) give the existence of a constant C such that

$$(\sigma r)\,\partial_{z_2} v^1 = C. \tag{4.33}$$

Since $\nabla u^1 = \partial_{z_1} v^1 f_1 + \partial_{z_2} v^1 f_2 + \partial_{z_3} v^1 e_3$, we have by (2.38)

$$\langle \partial_{z_2} v^1 \rangle = \langle \nabla u^1 \rangle \cdot f_2 = f_1 \cdot f_2 = 0. \tag{4.34}$$

By (4.26), since $0 < \alpha \leq \sigma \leq \beta$ and $r^{-1} \in L^1(Y)$, $(\sigma r)^{-1} \neq 0$ almost everywhere in $\mathbb{R}^2$. Combining (4.34) with (4.33), we get that

$$C\left\langle (\sigma r)^{-1} \right\rangle = 0, \tag{4.35}$$

hence, $C = 0$, which implies that v^1 is a function of z_1. On the other hand, the Alessandrini, Nesi [1] result combined with $v^1 = v^1(z_1)$ yields

$$\det(P_0) = \partial_{z_1} v^1 \partial_{z_2} v^2 - \partial_{z_1} v^2 \underbrace{\partial_{z_2} v^1}_{=0} = \partial_{z_1} v^1 \partial_{z_2} v^2 > 0 \quad \text{a.e. in } Y. \tag{4.36}$$

Moreover, by (2.37) we have

$$0 = \operatorname{div}\left(\sigma \nabla v^1\right) = \partial_{z_1}\left(\sigma\,\partial_{z_1} v^1\right) \quad \text{in } \mathscr{D}'(\mathbb{R}^3), \tag{4.37}$$

which implies that $\sigma\,\partial_{z_1} v^1$ is a function of z_2. By (4.36), we may define the two measurable functions f, g by

$$f(z_1) := \left(\partial_{z_1} v^1\right)^{-1} \quad \text{and} \quad g(z_2) := \sigma\,\partial_{z_1} v^1. \tag{4.38}$$

Therefore, we get that $\sigma(y') = f(z_1)\,g(z_2)$. In particular, f, g are h_i-periodic for $i = 1, 2$. Let us show that f, g, f^{-1}, g^{-1} are bounded functions. Denote $\delta := \max(|h_1|, |h_2|) > 0$. As $\alpha \le \sigma \le \beta$, we have by (4.38),

$$\beta^2 f^{-2}(z_1) \ge \sigma^2 f^{-2}(z_1) = g^2(z_2) \ge \alpha^2 f^{-2}(z_1) > 0 \quad \text{a.e. in } (z_1, z_2) \in (0, \delta)^2. \tag{4.39}$$

Integrating (4.39) successively with respect to z_1 and z_2 on $(0, \delta)$, we get that

$$\begin{cases} \alpha^{-2} C_1 \ge f^{-2}(z_1) \ge \beta^{-2} C_1, \\ \beta^2 \alpha^{-2} C_1 \ge g^2(z_2) \ge \alpha^2 \beta^{-2} C_1, \end{cases} \qquad \text{with } C_1 := \fint_0^\delta g^2(z_2)\, dz_2 > 0 \tag{4.40}$$

that is f, g, f^{-1}, g^{-1} are $L^\infty(\mathbb{R})$ functions. Integrating the inequality $\sigma(y') = f(z_1)\,g(z_2) \ge \alpha$ with respect to z_2 on $(0, \delta)$, we obtain that

$$f(z_1) \int_0^\delta g(z_2)\, dz_2 \ge \delta\,\alpha > 0 \quad \text{a.e. } z_1 \in \mathbb{R}, \tag{4.41}$$

that is f has a constant sign. Moreover, like in (4.34) we have

$$\langle f^{-1} \rangle = \langle \partial_{z_1} v^1 \rangle = \langle \nabla u^1 \rangle \cdot f_1 = f_1 \cdot f_1 = 1. \tag{4.42}$$

Hence f is a positive function, so is g by (4.38). Then, by a uniqueness argument the expression of σ implies that the potentials v^i, $i = 1, 2, 3$, are given by

$$\begin{cases} \partial_{z_1} v^1 = f^{-1}(z_1), & \partial_{z_2} v^1 = \partial_{z_3} v^1 = 0, \\ \partial_{z_2} v^2 = \langle g^{-1} \rangle^{-1} g^{-1}(z_2), & \partial_{z_1} v^2 = \partial_{z_3} v^2 = 0, \\ v^3 = z_3. \end{cases} \tag{4.43}$$

The conditions (4.32) and (4.36) give the existence of a constant C such that

$$r(y') = \frac{C}{\sigma\ \partial_{z_2} v^2} = C\,\frac{\langle g^{-1} \rangle}{f(z_1)}. \tag{4.44}$$

Using the expressions (4.30) and $|h'| = 1$, we obtain (4.27).

Conversely, if the conductivity and the Hall coefficient satisfy (4.27) with $\langle f^{-1} \rangle = 1$ (see Remark 4.1), the potentials v^i, for $i = 1, 2, 3$, are given by (4.43). Hence, it follows immediately (4.31) and thus the case of equality.

Third case: $h' \neq 0$ and $h_3 \neq 0$. Considering the third equality of (4.32), the first case shows that r is constant in Y. Moreover, taking into account the first and second components of (4.32), σ, r takes the form (4.27) by the second case. Hence, f is constant which gives (4.28). The converse is similar to the second case.

Case where $h_1 h_2 \neq 0$ and $h_1/h_2 \notin \mathbb{Q}$. As $h_1, h_2 \neq 0$, we are in the second or third case of the proof. Let $i \in \{1, 2\}$. We have proved that $u^i(y) = v^i(z)$ is a function of z_i. Moreover, $\varphi : y \mapsto u^i(y) - f_i \cdot y = v^i(z) - z_i$ is a function in $H^1_\sharp(Y; \mathbb{R})$. The function φ has a continuous representative and is h_j-periodic for $j = 1, 2$. As $h_1/h_2 \notin \mathbb{Q}$, φ is constant so is ∇v^i. Therefore, by (4.43), f, g are also constant. Finally by (4.27), σ and r are constant. $\square$

4.2.2 Four-phase checkerboard

In the section, we consider a four-phase checkerboard columnar structure. Let α_1, α_2, α_3, α_4 be positive numbers. Consider the Y-periodic conductivity only depending on $y' = (y_1, y_2)$, defined on the unit square $(-1/2, 1/2)^2$ by (see figure 4.1)

$$\sigma(y') = \begin{cases} \alpha_1 & \text{in } Q_1 := (0, 1/2)^2, \\ \alpha_2 & \text{in } Q_2 := (0, 1/2) \times (-1/2, 0), \\ \alpha_3 & \text{in } Q_3 := (-1/2, 0)^2, \\ \alpha_4 & \text{in } Q_4 := (-1/2, 0) \times (0, 1/2). \end{cases} \tag{4.45}$$

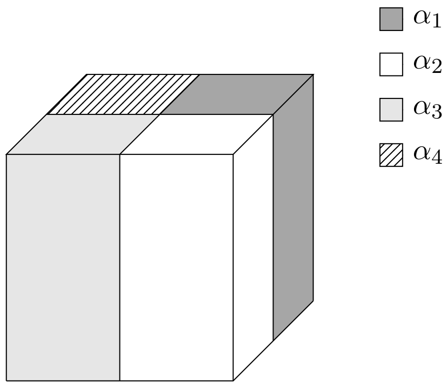

Figure 4.1: Period cell of the four-phase checkerboard columnar structure

We now state the following result:

Proposition 4.3. *Consider the conductivity defined by (4.45) and $\mathscr{D}(h, h)$ by (4.2) and assume that (4.26) is satisfied. Then, $\mathscr{D}(h, h) = 0$ if and only if one of the following conditions holds:*

- $h = 0$;
- $h \neq 0$ *is not parallel to e_i for $i = 1, 2, 3$, and σ, r are constant.*
- $h = h_3 e_3 \neq 0$, *and the Hall coefficient r is constant;*
- $h = h_i e_i \neq 0$ *for $i = 1, 2$, and there exists a constant C such that*

$$\alpha_1\, \alpha_3 = \alpha_2\, \alpha_4, \quad \textit{and} \quad r = C \left(\frac{\alpha_{6-2i}}{\alpha_1} \mathbb{1}_{\{y_i > 0\}} + \mathbb{1}_{\{y_i < 0\}} \right). \tag{4.46}$$

Remark 4.2. The case equality $\alpha_1\, \alpha_3 = \alpha_2\, \alpha_4$ corresponds to the case where the conductivity of the four-phase checkerboard is a tensor product of functions (see [15]).

When $\alpha_1\, \alpha_3 \neq \alpha_2\, \alpha_4$, Craster and Obnosov [10, 11] proved an intricate formula for the corrector P^0. In this case, σ is not a tensor product of functions which is consistent with Proposition 4.2.

Proof of Proposition 4.3. The case $(h_1, h_2) = (0, 0)$ and $h_3 \neq 0$ is a direct consequence of Proposition 4.2. Set, like in Proposition 4.2, $h' = (h_1, h_2)$.

First case: h is not parallel to e_i for $i = 1, 2, 3$. Assume that, without loss of generality, $|h'| = 1$, $h_i > 0$ for $i = 1, 2$. We apply Proposition 4.2. There exist two positive functions f and g in $L^\infty(\mathbb{R})$ which are h_i-periodic for $i = 1, 2$, and a constant C such that (4.27) holds. Since $\sigma(y') = f(z_1)\, g(z_2)$ (with the new variables (4.30)) is a piecewise constant function, $f(z_1)$ and $g(z_2)$ are constant in each open square Q_i for $i = 1, 2, 3, 4$. Considering the particular case of Q_1 and, for δ small enough, the rectangle

$$Q_{1,\delta} := \left\{ y' \in Q_1 \, : \, z_1 \in \left(\frac{h_1}{h_2}\delta, \frac{h_1 + h_2}{2} \right), \, z_2 \in (-\delta, \delta) \right\} \subset Q_1, \tag{4.47}$$

we get successively that

$$\begin{cases} f \text{ is constant on } I := \left(0, \dfrac{h_1 + h_2}{2} \right), \\ g \text{ is constant on } J := \left(-\dfrac{h_1}{2}, \dfrac{h_2}{2} \right). \end{cases} \tag{4.48}$$

Hence, σ is constant in the rectangle (see figure 4.2)

$$Q := \left\{ y' \in \left(-\frac{1}{2}, \frac{1}{2} \right) \, : \, (z_1, z_2) \in I \times J \right\}. \tag{4.49}$$

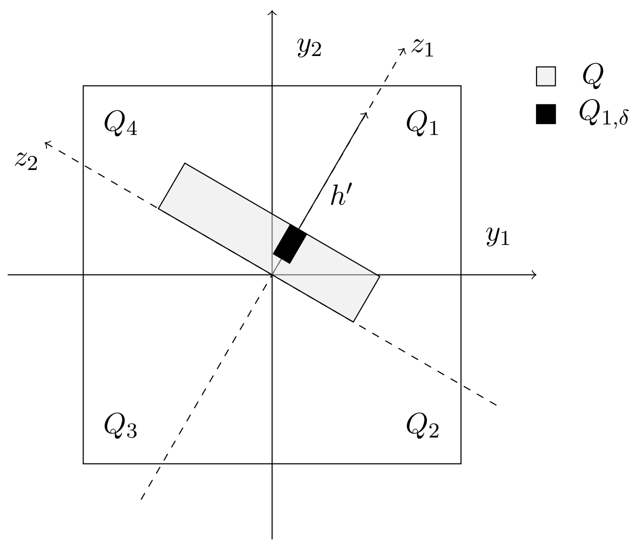

Figure 4.2: Cross-section of the period cell of the checkerboard in the (y_1, y_2)-plane

As Q intersects Q_1, Q_2, Q_4, σ is constant in $Q_1 \cup Q_2 \cup Q_4$, hence $\alpha_1 = \alpha_2 = \alpha_4$. Repeating the same argument with Q_3 in (4.48), we obtain that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. Therefore, σ, f, g are constant, so is r by (4.27).

Second case: $h = h_1 e_1 \neq 0$. Without loss of generality, assume that $h_1 = 1$. We again apply Proposition 4.2. There exist two positive functions f and g in $L^\infty(\mathbb{R})$ which are h_i-periodic for $i = 1, 2$, and a constant C' such that

$$\sigma(y') = f(y_1)\, g(y_2), \quad \text{and} \quad r(y') = \frac{C'}{f(y_1)} \quad \text{a.e. } y' \in (-1/2, 1/2)^2. \tag{4.50}$$

Since σ is piecewise constant in the four-phase checkerboard and f, g are respectively functions of the independent variables y_1, y_2, f, g are constant in each open square Q_i, for $i = 1, 2, 3, 4$. It follows immediately that there exist two positive constants C'_1 and C'_2 such that $C'_1\, C'_2 = \alpha_1^{-1}$ and

$$\begin{cases} f(y_1) & = C'_1 \left(\alpha_1 \mathbb{1}_{\{y_1 > 0\}} + \alpha_4 \mathbb{1}_{\{y_1 \leq 0\}} \right), \\ g(y_2) & = C'_2 \left(\alpha_1 \mathbb{1}_{\{y_2 > 0\}} + \alpha_2 \mathbb{1}_{\{y_2 \leq 0\}} \right), \end{cases} \quad \text{a.e. } y' \in (-1/2, 1/2)^2. \tag{4.51}$$

Finally, (4.51) and (4.50) imply that there exists a constant C such that

$$\alpha_1\, \alpha_3 = \alpha_2\, \alpha_4, \quad \text{and} \quad r(y') = C \left(\frac{\alpha_4}{\alpha_1} \mathbb{1}_{\{y_1 > 0\}} + \mathbb{1}_{\{y_1 < 0\}} \right). \tag{4.52}$$

Conversely, if (4.52) is satisfied, we can define f and g as in (4.51) with $C_1' C_2' = \alpha_1^{-1}$.

The case $h = h_2 e_2 \neq 0$ is quite similar. $\square$

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